GENERAL PROJECTIVE THEORY OF SPACE CURVES*

BY

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The theory of the invariants of a linear differential equation is not new. LAGUERRE, BRIOSCHI, HALPHEN, FORSYTH, BOUTON, FANO and others have written important papers on the subject. But in almost all of these papers the point of view is essentially analytical. It is only in the brilliant contributions of HALPHEN that one finds the idea of founding a geometry of curves upon this theory. But even HALPHEN's papers do not give the ground work for a thorough comprehension of the subject. This is to be found in the geometric interpretation of the semi-covariants which we shall discuss in this paper. With this as a basis, the whole theory becomes clear and transparent, and HALPHEN's results can easily be connected with it.

I have myself shown in recent years, how the general projective theory of ruled surfaces depends upon the theory of the invariants and covariants of a system of two linear differential equations of the second order. The present paper is governed by similar ideas, and to some extent depends upon this other theory. It is the main purpose of this paper to bring up the general projective theory of curves, based upon the theory of invariants, to the same level of perfection as the corresponding theory of ruled surfaces. Some of the theorems which we shall find, have of course been known for a long time. But even for most of these our proofs will be new.

§ 1. *The invariants and covariants.*

We shall confine our attention to the linear differential equation of the fourth order

$$y^{(4)} + 4p_1 y^{(3)} + 6p_2 y'' + 4p_3 y' + p_4 y = 0, \tag{1}$$

where

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2 y}{dx^2}, \quad \text{etc.}$$

If we make the transformations

$$y = \lambda(x)\bar{y}, \qquad \xi = \xi(x),$$

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where λ and μ are arbitrary functions, we obtain another equation of the same form as (1). The functions of the coefficients of (1), which remain invariant under these transformations, are the *invariants*. If they involve also the functions $y, y', y'', y^{(3)}$, they are called *covariants*.

To determine these invariant functions, it is best according to LIE's theory to make use of infinitesimal transformations. But this has been done in a number of papers by other authors and need not therefore be repeated. We shall, however, need the finite transformations for the later parts of this paper, and shall, therefore, write them down.

Consider first the simpler transformation

$$y = \lambda(x)\bar{y},$$

of the dependent variable alone. We shall speak of the corresponding invariant functions as *seminvariants* and *semi-covariants*. Let

$$\bar{y}^{(4)} + 4\pi_1\bar{y}^{(3)} + 6\pi_2\bar{y}'' + 4\pi_3\bar{y}' + \pi_4\bar{y} = 0$$

be the transformed equation. Then

$$\begin{aligned}
\pi_1 &= \frac{\lambda' + p_1\lambda}{\lambda}, \\
\pi_2 &= \frac{\lambda'' + 2p_1\lambda' + p_2\lambda}{\lambda}, \\
\pi_3 &= \frac{\lambda^{(3)} + 3p_1\lambda'' + 3p_2\lambda' + p_3\lambda}{\lambda}, \\
\pi_4 &= \frac{\lambda^{(4)} + 4p_1\lambda^{(3)} + 6p_2\lambda'' + 4p_3\lambda' + p_4\lambda}{\lambda},
\end{aligned} \tag{2}$$

whence one may deduce the absolute seminvariants

$$\begin{aligned}
P_2 &= p_2 - p_1' - p_1^2, \\
P_3 &= p_3 - p_1'' - 3p_1p_2 + 2p_1^3, \\
P_4 &= p_4 - 4p_1p_3 - 3p_2^2 + 12p_1^2p_2 - 6p_1^4 - p_1^{(3)},
\end{aligned} \tag{3}$$

and the relative semi-covariants, besides y which is obviously itself a semi-covariant,

$$\begin{aligned}
z &= y' + p_1y, \\
\rho &= y'' + 2p_1y' + p_2y, \\
\sigma &= y^{(3)} + 3p_1y'' + 3p_2y' + p_3y.
\end{aligned} \tag{4}$$

The absolute semi-covariants are z/y, ρ/y, σ/y. All other semi-covariants and seminvariants are functions of these and of the derivatives of P_2, P_3, P_4.

From (4) we deduce the following equations, which we shall use later:

$$
\begin{aligned}
y' &= -p_1 y + z,\\
z' &= -P_2 y - p_1 z + \rho,\\
\rho' &= -(P_3 - P_2')y - 2P_2 z - p_1\rho + \sigma,\\
\sigma' &= -(P_4 - P_3')y - 3(P_3 - P_2')z - 3P_2\rho - p_1\sigma,
\end{aligned}
\tag{5}
$$

and also

$$
\begin{aligned}
y'' &= (2p_1^2 - p_2)y - 2p_1 z + \rho,\\
y^{(3)} &= (-p_3 + 6p_1p_2 - 6p_1^3)y + (-3p_2 + 6p_1^2)z - 3p_1\rho + \sigma,\\
y^{(4)} &= (-p_4 + 8p_1p_3 - 36p_1^2p_2 + 6p_2^2 + 24p_1^4)y\\
&\qquad + (-4p_3 + 24p_1p_2 - 24p_1^3)z + (-6p_2 + 12p_1^2)\rho - 4p_1\sigma.
\end{aligned}
\tag{6}
$$

We now proceed to make a transformation of the independent variable $\xi = \xi(x)$. We find, denoting the coefficients of the transformed equation by $\bar{p}_k$,

$$
\begin{aligned}
\bar{p}_1 &= \frac{1}{\xi'}(p_1 + \tfrac{3}{2}\eta),\\
\bar{p}_2 &= \frac{1}{(\xi')^2}[p_2 + 2\eta p_1 + \tfrac{1}{6}(4\mu + 9\eta^2)],\\
\bar{p}_3 &= \frac{1}{(\xi')^3}[p_3 + \tfrac{3}{2}\eta p_2 + (\mu + \tfrac{3}{2}\eta^2)p_1 + \tfrac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)],\\
\bar{p}_4 &= \frac{1}{(\xi')^4}p_4,
\end{aligned}
\tag{7}
$$

where we have put

$$
\eta = \frac{\xi''}{\xi'}, \qquad \mu = \eta' - \tfrac{1}{2}\eta^2. \tag{8}
$$

We find further

$$
\begin{aligned}
\bar{z} &= \frac{1}{\xi'}(z + \tfrac{3}{2}\eta y),\\
\bar{\rho} &= \frac{1}{(\xi')^2}[\rho + 2\eta z + \tfrac{1}{6}(4\mu + 9\eta^2)y],\\
\bar{\sigma} &= \frac{1}{(\xi')^3}[\sigma + \tfrac{3}{2}\eta\rho + (\mu + \tfrac{3}{2}\eta^2)z + \tfrac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)y],
\end{aligned}
\tag{9}
$$

cogredient with (7).

Making use of these equations, we find

$$P_2 = \frac{1}{(\xi')^2}[P_2 - \tfrac{5}{6}\mu],$$

$$P_3 = \frac{1}{(\xi')^3}[P_3 - 3\eta P_2 - \tfrac{5}{4}\mu' + \tfrac{10}{4}\eta\mu], \tag{10}$$

$$\overline{P}_4 = \frac{1}{(\xi')^4}[P_4 - 6\eta P_3 - 4\eta' P_2 + 11\eta^2 P_2 + \tfrac{15}{2}\eta\mu' - \tfrac{15}{2}\eta^2\mu - \tfrac{3}{2}\mu'' + \tfrac{19}{6}\mu^2],$$

whence

$$\overline{P}_2' = \frac{1}{(\xi')^3}[P_2' - 2\eta P_2 - \tfrac{5}{6}\mu' + \tfrac{5}{3}\eta\mu],$$

$$\overline{P}_2'' = \frac{1}{(\xi')^4}[P_2'' - 5\eta P_2' - 2\mu P_2 + 5\eta^2 P_2 - \tfrac{5}{6}\mu'' + \tfrac{5}{3}\mu^2 - \tfrac{25}{6}\mu\eta^2 + \tfrac{25}{6}\mu'\eta], \tag{11}$$

$$\overline{P}_3' = \frac{1}{(\xi')^4}[P_3' - 3\eta(P_2' + P_3) - 3\mu P_2 + \tfrac{15}{2}\eta^2 P_2 - \tfrac{5}{4}\mu'' + \tfrac{25}{4}\eta\mu' - \tfrac{25}{4}\mu\eta^2 + \tfrac{10}{4}\mu^2].$$

We find therefore the following invariants and covariants:

$$\theta_3 = P_3 - \tfrac{3}{2}P_2', \qquad \theta_4 = P_4 - 2P_3' + \tfrac{6}{5}P_2'' - \tfrac{6}{25}P_2^2,$$

$$\theta_{3\cdot 1} = 6\theta_3\theta_3'' - 7\theta_3'^2 - \tfrac{108}{5}P_2\theta_3^2,$$

$$C_2 = 10z^2 - 15y\rho - 12P_2y^2, \tag{12}$$

$$C_3 = 10z^3 - 3C_2z - 9(5\sigma + 6P_2z + P_3y)y^2,$$

$$C_4 = 2\theta_3 z + \theta_3' y,$$

where the index indicates the weight. In denoting one invariant of weight 8 by $\theta_{3\cdot 1}$ we follow a notation due to FORSYTH. An invariant may be regarded as a covariant of degree zero. With this understanding, it suffices to say that the effect of the complete transformation

$$\bar{y} = \lambda(x)y, \qquad \bar{\xi} = \xi(x),$$

upon a covariant of degree d and of weight w, is to transform it into $\overline{C}$, where

$$\overline{C} = \frac{\lambda^d}{(\xi')^w}C.$$

LIE'S theory shows that all other invariants and covariants may be deduced from these by algebraic and differentiation processes.

§ 2. *Canonical forms.*

Equations (2) show that if we make the transformation

$$y = e^{-\int p_1 dx} \bar{y},$$

the coefficients of the resulting equation for $\bar{y}$ will be

$$\pi_1 = 0, \qquad \pi_2 = P_2, \qquad \pi_3 = P_3, \qquad \pi_4 = P_4 + 3P_2^2.$$

We shall say that the equation has been put into the *semi-canonical* form.

From (10) we see that if $\xi(x)$ be chosen so that

$$\eta' - \tfrac{1}{2}\eta^2 = \mu = \tfrac{6}{5}P_2, \tag{13}$$

in the resulting equation $\bar{P}_2$ will be zero. Since P_2 is a seminvariant, any transformation of the form $\bar{y} = \lambda y$ will not disturb the equation $P_2 = 0$, and we may again choose λ so as to make the coefficient of $d^3\bar{y}/d\xi^3$ vanish. It is therefore always possible to reduce the equation to the form

$$\frac{d^4\bar{y}}{d\xi^4} + 4\pi_3 \frac{d\bar{y}}{d\xi} + \pi_4 \bar{y} = 0,$$

which we shall call the LAGUERRE-FORSYTH *canonical form.* This is equivalent to assuming $p_1 = p_2 = 0$ in the original equation.

If $\theta_3 \neq 0$, we may transform the independent variable so as to make $\bar{\theta}_3 = 1$. In fact we have for an arbitrary transformation

$$\bar{\theta}_3 = \frac{1}{(\xi')^3}\theta_3.$$

If, therefore, we put

$$\xi = \int \sqrt[3]{\theta_3}\, dx, \tag{14}$$

$\bar{\theta}_3$ will be equal to unity. We may again by a transformation of the form $\bar{y} = \lambda y$ make p_1 vanish. The canonical form which is characterized by the conditions

$$p_1 = 0, \qquad \theta_3 = 1,$$

we may properly denote as the HALPHEN *canonical form.*

In our geometrical discussions only the quantity

$$\eta = \frac{\xi''}{\xi'},$$

not ξ itself will be of any importance. λ also is an unimportant factor which has no geometrical significance. Equation (13) shows, therefore, that the reduction to the LAGUERRE-FORSYTH form can always be accomplished in ∞^1 essentially different ways. It is important to remark that (13) is an equation of the RICCATI form, so that the cross-ratio of any four solutions is constant.

The HALPHEN form on the other hand can be obtained in just one way, if it exists at all, i. e., if $\theta_3 \neq 0$. If θ_3 vanishes, θ_4 may be reduced to unity unless it also is equal to zero. The case when both θ_3 and θ_4 vanish, is especially simple. The LAGUERRE-FORSYTH form reduces to

$$\frac{d^4\bar{y}}{d\bar{\xi}^4} = 0.$$

If two equations of the form (1) can be transformed into each other by a transformation of the kind here considered, we shall call them equivalent. Clearly, for equivalent equations, the corresponding absolute invariants are equal.

If equation (1) is given, the invariants θ_3, θ_4, $\theta_{3\cdot 1}$, etc., are known functions of x. Conversely, equations (12) show that if θ_3, θ_4, $\theta_{3\cdot 1}$ are given as arbitrary functions of x, provided that $\theta_3 \neq 0$, P_2, P_3 and P_4 are determined uniquely. If $\theta_3 = 0$, then $\theta_{3\cdot 1} = 0$ also, and we must assign a further condition. The function

$$\theta_{4\cdot 1} = 8\theta_4\theta_4'' - 9(\theta_4')^2 - \frac{80}{3}P_2\theta_4^2 \tag{15}$$

is also an invariant. If $\theta_3 = 0$, and θ_4, $\theta_{4\cdot 1}$ are given, P_2, P_3, P_4 are determined uniquely. If both θ_3 and θ_4 vanish, all invariants are zero, and the equation may be reduced to the form

$$\frac{d^4\eta}{d\bar{\xi}^4} = 0.$$

As we may always assume that $p_1 = 0$, we see that *the differential equation* (1) *is essentially determined when its invariants are given as functions of* x.

The LAGRANGE adjoint of (1) is

$$\begin{aligned} u^{(4)} - 4p_1u^{(3)} + 6(p_2 - 2p_1')u'' - 4(p_3 - 3p_2' + 3p_1'')u' \\ + (p_4 - 4p_3' + 6p_2'' - 4p_1^{(3)})u = 0. \end{aligned} \tag{16}$$

If $y_1, \cdots, y_4$ constitute a fundamental system of (1), the minors of $x_1, \cdots, x_4$ in the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \end{vmatrix}$$

multiplied by a common factor, which does not interest us, form a fundamental system of (16).

If we denote the seminvariants of (16) by Π_2, Π_3, Π_4, we have

$$\Pi_2 = P_2, \qquad \Pi_3 = -P_3 + 3P_2', \qquad \Pi_4 = P_4 - 4P_3' + 6P_2'', \tag{17}$$

whence follows reciprocally

$$P_2 = \Pi_2, \qquad P_3 = -\Pi_3 + 3\Pi_2', \qquad P_4 = \Pi_4 - 4\Pi_3' + 6\Pi_2''.$$

The invariants of (16) differ from the invariants of (1) only in this that the sign of θ_3 is changed.

§ 3. *Geometrical interpretation.*

If the functions $y_1, \cdots, y_4$ constitute a fundamental system of (1) we may interpret them as the homogeneous coördinates of a point P_y of a curve C_y in ordinary space. The coefficients p_1, p_2, p_3, p_4 of (1) are invariants of the general projective group. The transformation $y = \lambda\bar{y}$ does not change the ratios $y_1 : y_2 : y_3 : y_4$, and therefore leaves the curve C_y invariant. The transformation $\xi = \xi(x)$ merely changes the parameter in terms of which the coördinates are expressed. It is clear therefore that any system of equations invariant under these transformations expresses a projective property of the curve C_y in the vicinity of the point P.

The LAGRANGE adjoint of (1) may be taken to represent the same curve in tangential coördinates, or else a reciprocal curve in point coördinates.

We may therefore state the results of § 2 as follows: *If the invariants of a curve are given as functions of x, the curve is determined except for projective transformations. If the invariants of two curves, except those of weight three, are respectively equal to each other, while the invariants of weight three differ only in sign, the two curves are dualistic to each other. Those curves are self-dual for which $\theta_3 = 0$.*

Moreover these latter curves are the only curves which are self-dual in the restricted sense that a dualistic transformation exists which converts every point of the curve into the tangent plane of that point, and vice versâ, while every tangent is converted into itself.

If we put $y = y_k (k = 1, 2, 3, 4)$ into the expressions for z, ρ, σ we obtain three other points P_z, P_ρ, P_σ, which describe curves C_z, C_ρ, C_σ as x varies, curves which are closely connected with C_y. P_z is clearly a point on the tangent of C_y constructed at P_y; P_ρ is in the plane osculating C_y at P_y, while P_σ is outside of this plane. These four points are never coplanar except at those exceptional points of C_y whose osculating planes are stationary, i. e., have more than three consecutive points in common with the curve.

In order to study the curve C_y in the vicinity of P_y, it will therefore be convenient to introduce the tetrahedron $P_y P_z P_\rho P_\sigma$ as tetrahedron of reference, with the further convention that if any expressions of the form

$$u_k = \alpha_1 y_k + \alpha_2 z_k + \alpha_3 \rho_k + \alpha_4 \sigma_k \qquad (k = 1, 2, 3, 4)$$

offer themselves the coördinates of the corresponding point P_u shall be

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

In writing u_k the index k may be suppressed, so that a single expression

$$\alpha_1 y + \alpha_2 z + \alpha_3 \rho + \alpha_4 \sigma$$

represents the point $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in this special system of coördinates.

As the independent variable x is changed, the tetrahedron of reference is changed in accordance with equations (9). P_y of course remains the same; P_z is changed into $P_{\bar{z}}$, which may obviously be any point on the tangent: etc. Thus while an arbitrary transformation of the parameter x does not affect the curve C_y itself, it does very materially affect the semi-covariant curves C_z, C_ρ and C_σ. It is clear however that two transformations $\xi = \xi(x)$, for which $\eta = \xi''/\xi'$ has the same value, are geometrically equivalent. We may also, without affecting the position of the points P_z, P_ρ, P_σ, assume that (1) is written in the semi-canonical form, so that $p_1 = 0$. For, in order to put (1) into the semi-canonical form, we need only multiply y by a certain factors λ, which will then also appear multiplied into the semi-covariants z, ρ and σ.

Let us then assume $p_1 = 0$. We shall have $z = y'$. If we differentiate (1) and eliminate y between the resulting equation and (1), we shall find

$$\begin{aligned}(18)\qquad &(P_4 + 3P_2^2)z^{(4)} - (P_4' + 6P_2P_2')z^{(3)} + 6P_2(P_4 + 3P_2^2)z'' \\ &+ [(6P_2' + 4P_3)(P_4 + 3P_2^2) - 6P_2(P_4' + 6P_2P_2')]z' \\ &+ [(4P_3' + P_4 + 3P_2^2)(P_4 + 3P_2^2) - 4P_3(P_4' + 6P_2P_2')]z = 0,\end{aligned}$$

if $P_4 + 3P_2^2 \neq 0$. If $P_4 + 3P_2^2 = 0$ we find

$$(19)\qquad z^{(3)} + 6P_2z' + 4P_3z = 0.$$

Equation (18) determines the curve C_z in the same way as (1) determines C_y. But if $P_4 + 3P_2^2 = 0$, z satisfies (19) showing that the curve C_z is in this case a plane curve. Therefore, if the variable ξ be so chosen as to make $P_4 + 3P_2^2 = 0$, the corresponding curve $C_{\bar{z}}$ is a plane section of the developable surface whose cuspidal edge is C_y. In harmony with this, equations (10) show that the most general value of η, which satisfies the condition $P_4 + 3P_2^2 = 0$, contains three arbitrary constants, as it should since there are ∞^3 planes in space.

We shall need to consider the ruled surfaces generated by those edges of our tetrahedron which meet in P_y. Of these we know one immediately, namely the developable which has C_y as its edge of regression, and of which P_yP_z is a generator. The ruled surface generated by P_yP_ρ clearly has C_y as an asymptotic curve; for, the plane $P_yP_\rho P_z$ is both osculating plane of C_y at P_y, and

tangent plane of the surface at P_y. If we assume $p_1 = 0$, this ruled surface may be studied by means of the equations

$$\begin{aligned} &y'' + P_2 y - \rho = 0, \\ &\rho'' + (4P_3 - 2P_2')y' + (P_4 - P_2'' - 2P_2^2)y + 5P_2\rho = 0, \end{aligned} \tag{20}$$

in accordance with the general theory of ruled surfaces as developed in former papers of the author. To prove (20) we need only differentiate twice the expression for ρ, express $y^{(3)}$ and $y^{(4)}$ in terms of y, z, ρ, σ, and eliminate z and σ.

The ruled surface generated by $P_y P_\sigma$ is especially important. We have,

$$\sigma = y^{(3)} + 3p_1 y^{(2)} + 3p_2 y' + p_3 y, \tag{21}$$

whence

$$\begin{aligned} \sigma' &= y^{(4)} + 3p_1 y^{(3)} + (3p_1' + 3p_2)y'' + (3p_2' + p_3)y' + p_3' y, \\ \sigma'' &= y^{(5)} + 3p_1 y^{(4)} + 3(2p_1' + p_2)y^{(3)} \\ &\quad + (3p_1'' + 6p_2' + p_3)y'' + (3p_2'' + 2p_3')y' + p_3'' y. \end{aligned} \tag{22}$$

From (21) we find

$$\begin{aligned} y^{(3)} &= \sigma - 3p_1 y'' - 3p_2 y' - p_3 y, \\ y^{(4)} &= \sigma' - 3p_1\sigma - 3(p_1' + p_2 - 3p_1^2)y'' \\ &\quad - (3p_2' + p_3 - 9p_1 p_2)y' - (p_3' - 3p_1 p_3)y. \end{aligned} \tag{23}$$

If we substitute these values in (1) we obtain the equation

$$\begin{aligned} &3(p_2 - p_1' - p_1^2)y'' + 3(p_3 - p_2' - p_1 p_2)y' + \sigma' \\ &\quad + (p_4 - p_1 p_3 - p_3')y + p_1\sigma = 0, \end{aligned} \tag{24}$$

where the coefficient of y'' is $3P_2$.

Let us differentiate both members of this equation, and eliminate y'' and $y^{(3)}$ by means of (23) and (24). We shall find

$$\begin{aligned} 3P_2\sigma'' &= (q_3 r_3 + 3P_2 q_4)y' + (3P_2 q_1 - q_3)\sigma' \\ &\quad + (r_4 q_3 + 3P_2 q_5)y + (3P_2 q_2 - p_1 q_3)\sigma, \end{aligned} \tag{25}$$

where

$$\begin{aligned} q_1 &= -p_1, \qquad q_2 = 2p_1' - 3p_2 + 3p_1^2, \qquad q_3 = -3P_3 + 3p_1 P_2, \\ q_4 &= -2p_3' + 3p_2'' + 3p_1 p_2' - 6p_1' p_2 - p_4 + p_1 p_3 + 9p_2^2 - 9p_1^2 p_2, \\ q_5 &= -(p_4' - p_3'' - p_1 p_3' - p_1' p_3) + 3p_3(p_2 - p_1' - p_1^2), \\ r_3 &= -3(p_3 - p_2' - p_1 p_2), \\ r_4 &= -(p_4 - p_3' - p_1 p_3). \end{aligned} \tag{26}$$

Equations (24) and (25) define the ruled surface generated by $P_y P_\sigma$. If we assume $p_1 = 0$, we find

$$\begin{aligned} y'' + p_{11}y' + p_{12}\sigma' + q_{11}y + q_{12}\sigma &= 0, \\ \sigma'' + p_{21}y' + p_{22}\sigma' + q_{21}y + q_{22}\sigma &= 0, \end{aligned} \tag{27}$$

where

$$\begin{aligned} p_{11} &= \frac{P_3 - P_2'}{P_2}, \qquad p_{12} = \frac{1}{3P_2}, \qquad q_{11} = \frac{P_4 + 3P_2^2 - P_3'}{3P_2}, \qquad q_{12} = 0, \\ p_{21} &= \frac{1}{P_2}\left[-3P_3^2 + 3P_3P_2' + 2P_2P_3' - 3P_2P_2'' + P_2P_4 - 6P_2^3\right], \\ p_{22} &= -\frac{P_3}{P_2}, \\ q_{21} &= \frac{1}{P_2}\left[-P_3P_4 - 6P_2^2P_3 + P_3P_3' + P_2P_4' + 6P_2^2P_2' - P_2P_3''\right], \\ q_{22} &= 3P_2. \end{aligned} \tag{28}$$

If (1) is written in the LAGUERRE-FORSYTH form, $P_2 = 0$. In that case the two equations (27) reduce to the single equation

$$P_3y' + \tfrac{1}{3}\sigma' + \tfrac{1}{3}(P_4 - P_3')y = 0, \tag{29}$$

which proves that *in this case the surface generated by $P_y P_\sigma$ is developable.* For in this case the tangents constructed respectively to C_y at P_y and to C_σ at P_σ are coplanar. Moreover, only if $P_2 = 0$ will the surface generated by $P_y P_\sigma$ be a developable.

Let

$$\tau = \lambda y + \mu\sigma$$

represent the point at which $P_y P_\sigma$ intersects the edge of regression of the developable. Then, since $P_y P_\sigma$ must be tangent to the edge of regression, we shall have $\tau' = \alpha y + \beta\sigma$, or

$$(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' + \mu\sigma' = 0.$$

But according to (29)

$$\sigma' = -(P_4 - P_3')y - 3P_3y',$$

so that

$$(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' - \mu\left[(P_4 - P_3')y + 3P_3y'\right] = 0,$$

where for y' we could also write z. Such a relation between P_y, P_z, P_σ would, however, make these three points collinear, and, therefore, $P_y, P_z, P_\rho, P_\sigma$ coplanar, unless all of the coefficients are zero. We have seen, however, that this

can happen only for such points P_y at which the osculating plane is stationary. We must have, therefore,

$$\lambda' - \alpha - \mu(P_4 - P_3') = 0, \qquad \mu' - \beta = 0, \qquad \lambda - 3\mu P_3 = 0,$$

whence

$$\lambda = 3\mu P_3, \qquad \beta = \mu', \qquad \alpha = \lambda' - \mu(P_4 - P_3').$$

We see, therefore, that

(30) $$\tau = 3P_3 y + \sigma$$

represents the edge of regression of the developable to which the ruled surface generated by $P_y P_\sigma$ reduces when $P_2 = 0$.

If $p_1 = 0$ and $P_2 = 0$, equations (2) and (10) show that the most general transformations of the variables, which do not disturb these conditions, satisfy the equations

$$\frac{\lambda'}{\lambda} + \tfrac{3}{2}\frac{\xi''}{\xi'} = 0, \qquad \mu = \eta' - \tfrac{1}{2}\eta^2 = 0,$$

which give on integration

$$\lambda = \frac{C}{(\xi')^{\frac{3}{2}}}, \qquad \eta = \frac{-2c}{1+cx}.$$

If we transform τ under this assumption, we find that it is converted into

(30a) $$\bar{\tau} = \frac{1}{(\xi')^3 \lambda}\left[\sigma + \tfrac{3}{2}\eta\rho + \tfrac{3}{2}\eta^2 z + (\tfrac{3}{4}\eta^3 + 3P_3)y\right],$$

where η may have any numerical value.

Let us recapitulate. *The ruled surfaces generated by $P_y P_{\bar{\sigma}}$ are infinite in number. Their general expression involves an arbitrary function η. Among these surfaces there exists a single infinity of developables. If $P_2 = 0$, the surface generated by $P_y P_\sigma$ is one of these, and the locus of P_τ is its edge of regression, where*

(30) $$\tau = 3P_3 y + \sigma,$$

P_τ being the point where $P_y P_\sigma$ intersects the edge of regression. If we construct all the ∞^1 lines $P_y P_{\bar{\sigma}}$ through P_y, which are generators of the above mentioned family of developables, and mark upon each of them the point $P_{\bar{\tau}}$ where it intersects the cuspidal edge of the developable to which it belongs, the locus of these points is a twisted cubic curve. The equations of this curve referred to a parameter η and to the fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ are

(31) $$x_1 = 3P_3 + \tfrac{3}{4}\eta^3, \qquad x_2 = \tfrac{3}{2}\eta^2, \qquad x_3 = \tfrac{3}{2}\eta, \qquad x_4 = 1.$$

We shall see later that this cubic has five consecutive points in common with the curve C_y at P_y, i. e., that it has at this point with C_y a contact of the fourth

order. We shall speak of it as the *torsal cubic of* P_y, on account of its connection with the developables which we have just been considering.

Equations (31) give the parametric equations of the torsal cubic referred to a special tetrahedron of reference for which $P_2 = 0$. We shall need its equations in a more general form. These may be easily obtained. Consider the expression

$$\lambda = (3\theta_3 + \tfrac{3}{10}P_2' + \tfrac{6}{5}P_2 t + \tfrac{3}{4}t^3)y + (\tfrac{6}{5}P_2 + \tfrac{3}{2}t^2)z + \tfrac{3}{2}t\rho + \sigma \tag{32}$$

in which t may for the moment be regarded as a parameter independent of x. Denote by $\bar{\lambda}$ the corresponding expression formed from the quantities $\bar{P}_2$, $\bar{P}_3$, etc., $\bar{y}$, $\bar{z}$, $\bar{\rho}$, $\bar{\sigma}$ after the general transformation $\xi = \xi(x)$. We shall find that $(\xi')^3\bar{\lambda}$ is equal to λ after t has been replaced by $t\xi' + \eta = t_1$. But of course this transformation may be chosen so as to make $\bar{P}_2 = 0$, which would make $\bar{\lambda}$ identical with $\bar{\tau}$ except for the notation.

We see, therefore, that the expression λ, *or the equations*

$$\begin{aligned} x_1 &= 3\theta_3 + \tfrac{3}{10}P_2' + \tfrac{6}{5}P_2\eta + \tfrac{3}{4}\eta^3, \\ x_2 &= \tfrac{6}{5}P_2 + \tfrac{3}{2}\eta^2, \qquad x_3 = \tfrac{3}{2}\eta, \qquad x_4 = 1, \end{aligned} \tag{33}$$

represent the torsal cubic referred to the fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ *when this is chosen in as general a way as is compatible with its definition.*

If in (32) t is chosen as a function of x, as x varies we obtain a curve on the surface formed by the totality of torsal cubics. If in particular t satisfies as function of x the differential equation

$$t' - \tfrac{1}{2}t^2 = \tfrac{6}{5}P_2,$$

we obtain the cuspidal edge of one of the developables.

§ 4. *The osculating cubic, conic and linear complex.*

A space cubic is determined by six of its points provided that no four of these points are coplanar. If, therefore, we take upon C_y, besides P_y, five other points, we shall in general obtain a perfectly definite space cubic determined by these six points. As these points approach coincidence with P_y, this cubic will in general approach a limit, which shall be called the *osculating cubic*. We proceed to find its equations.

Let P_y correspond to the value of $x = a$, which we shall suppose is an ordinary point for our differential equation. Then y may be developed by Taylor's theorem into a series proceeding according to powers of $x - a$. By putting $x - a = x'$ the development will be in powers of x'. We may therefore assume in the first place that $a = 0$. Let us assume, further, that $p_1 = 0$ and $P_2 = 0$. Then we shall have from (5) and (6),

$$(34)\qquad \begin{aligned} &y' = z, \quad y'' = \rho, \quad y^{(3)} = -P_3 y + \sigma, \quad y^{(4)} = -P_4 y - 4P_3 z, \\ &y^{(5)} = -P_4' y - (P_4 + 4P_3')z - 4P_3\rho. \end{aligned}$$

In accordance with the definition of our coördinates, we denote the coefficients of y, z, ρ, σ in this expansion carried as far as x^5, by y_1, y_2, y_3, y_4. We may of course multiply these quantities by a common factor since the coördinates are homogeneous. We shall multiply by 120 so as to clear of fractions. This gives

$$(35)\qquad \begin{aligned} y_1 &= 120 - 20P_3 x^3 - 5P_4 x^4 - P_4' x^5 + \cdots, \\ y_2 &= 120x - 20P_3 x^4 - (4P_3' + P_4)x^5 + \cdots, \\ y_3 &= 60x^2 - 4P_3 x^5 + \cdots, \\ y_4 &= 20x^3 + \cdots. \end{aligned}$$

We see at once that the following equations are exact up to terms no higher than the fifth order:

$$(36)\qquad 3y_2 y_4 - 2y_3^2 = 0, \qquad 5(2y_1 y_3 - y_2^2) - 6P_3 y_3 y_4 = 0.$$

These equations must be satisfied by the coördinates of any point of the osculating cubic, since this must have contact of the fifth order with C_y at P_y. *They are therefore its equations, referred to this special tetrahedron of reference. In terms of a parameter t we may write*

$$(37)\qquad x_1 = 15 + 12P_3 t^3, \qquad x_2 = 30t, \qquad x_3 = 30t^2, \qquad x_4 = 20t^3.$$

The equation

$$3x_2 x_4 - 2x_3^2 = 0$$

is that of a cone whose vertex is P_y and which contains the osculating cubic. It may also be obtained by determining that cone of the second order with its vertex at P_y which has the closest possible contact with C_y, viz. contact of the fifth order. We shall speak of it as the *osculating cone*. We notice at once that the torsal cubic also lies upon the osculating cone. This is shown by equations (31) which are referred to the same system of coördinates as that employed here. If we put $\eta = 1/t$ in (31) and if we multiply by $20t^3$, (31) becomes

$$(31a)\qquad x_1 = 15 + 60P_3 t^3, \qquad x_2 = 30t, \qquad x_3 = 30t^2, \qquad x_4 = 20t^3,$$

which differs from (37) only in having $5P_3$ in place of P_3.

By a method of reasoning precisely similar to that of the last paragraph, we find that *the expression*

$$(38)\qquad (12P_3 - 12P_2' + 24P_2\tau + 15\tau^3)y + 20(\tfrac{6}{5}P_2 + \tfrac{3}{2}\tau^2)z + 30\tau\rho + 20\sigma$$

represents an arbitrary point of the osculating cubic when the tetrahedron of reference is not restricted to the condition $P_2 = 0$. For this expression remains invariant under the general transformation $\xi = \xi(x)$, and reduces to (37) for $P_2 = 0$, if $\tau = 1/t$.

The equation of the plane, which osculates the osculating cubic at the point whose parameter is τ, turns out to be

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0,$$

where

$$(39) \qquad \begin{aligned} u_1 &= -20, \qquad u_2 = 30\tau, \qquad u_3 = 16P_2 - 30\tau^2, \\ u_4 &= 12P_3 - 12P_2' - 36P_2\tau + 15\tau^3. \end{aligned}$$

For every value of τ this intersects the osculating plane, $x_4 = 0$, in a straight line

$$-20x_1 + 30\tau x_2 + (16P_2 - 30\tau^2)x_3 = 0.$$

The envelope of these lines will be obtained by eliminating τ between this equation and that obtained from it by partial differentiation with respect to τ; the latter equation is

$$30x_2 - 60\tau x_3 = 0.$$

We thus find

$$(40) \qquad -40x_1x_3 + 15x_2^2 + 32P_2x_3^2 = 0,$$

the equation of the osculating conic, which may be defined as a part of the intersection of the developable of the osculating cubic with the osculating plane. The other part of this intersection is the tangent, which must be counted twice.

It is not without interest to verify that (37) represents the osculating cubic by another method. We have from (35) as non-homogeneous coördinates of the points of C_y in the vicinity of P_y

$$(41) \qquad \frac{y_2}{y_1} = x - \tfrac{1}{120}(P_3' - 4P_4)x^5 + \cdots, \qquad \frac{y_3}{y_1} = \tfrac{1}{2}x^2 + \tfrac{1}{20}P_3x^5 + \cdots,$$

$$\frac{y_4}{y_1} = \tfrac{1}{6}x^3 + \cdots.$$

From (37) we find for the points of the osculating cubic

$$\frac{x_2}{x_1} = 2t - \tfrac{8}{5}P_3t^4 + \cdots, \qquad \frac{x_3}{x_1} = 2t^2 - \tfrac{8}{5}P_3t^5 + \cdots, \qquad \frac{x_4}{x_1} = \tfrac{4}{3}t^3 + \cdots.$$

If we put

$$t = \tfrac{1}{2}x[1 + \tfrac{1}{10}P_3x^3 - \tfrac{1}{30}(P_3' + P_4)x^4 + \cdots],$$

these two expansions coincide up to terms of the fifth order. For the torsal cubic we have according to (31a),

$$\frac{x_2}{x_1} = 2t - 8P_3t^4\cdots, \qquad \frac{x_3}{x_1} = 2t^2 - 8P_3t^5 + \cdots, \qquad \frac{x_4}{x_1} = \tfrac{4}{3}t^3 + \cdots.$$

If we put into these equations

$$t = \tfrac{1}{2}x(1 + ax + bx^2 + cx^3 + dx^4 + \cdots),$$

we find that these expansions will agree with (41) up to terms of the fourth order for

$$a = 0, \qquad b = 0, \qquad c = \tfrac{1}{2}P_3.$$

but that it is impossible to make them agree with (41) any farther unless $P_3 = 0$. *In general, therefore, the torsal cubic has with C_y a contact of the fourth order. Only if $\theta_3 = 0$ may the order of contact be higher. In that case the torsal and osculating cubics coincide.*

We proceed to deduce the equation of the *osculating linear complex*, i. e., of that linear complex determined by five consecutive tangents of the curve. We assume again $p_1 = 0$ and $P_2 = 0$. Denote by Y and Z the expansions of y and z in the vicinity of P_y. Then we have up to terms of the fourth order

$$Y = y(1 - \tfrac{1}{6}P_3x^3 - \tfrac{1}{24}P_4x^4) + z(x - \tfrac{1}{6}P_3x^4) + \tfrac{1}{2}\rho x^2 + \tfrac{1}{6}\rho x^3,$$

$$Z = y(-\tfrac{1}{2}P_3x^2 - \tfrac{1}{6}P_4x^3 - \tfrac{1}{24}P_4'x^4) + z[1 - \tfrac{2}{3}P_3x^3 - \tfrac{1}{24}(4P_3' + P_4)x^4] + \rho(x - \tfrac{1}{6}P_3x^4) + \sigma\tfrac{1}{2}x^2.$$

If we denote the coefficients of y, z, ρ, σ in these two expressions by $y_1, \cdots, y_4$ and $z_1, \cdots, z_4$, respectively, the Plückerian line-coördinates of the tangent will be

$$\omega_{ik} = y_iz_k - y_kz_i,$$

whence

$$\omega_{12} = 1 - \tfrac{1}{6}P_3x^3 + \tfrac{1}{8}P_4x^4 + \cdots, \quad \omega_{13} = x - \tfrac{1}{12}P_3x^4 + \cdots, \quad \omega_{14} = \tfrac{1}{2}x^2 + \cdots,$$

$$\omega_{23} = \tfrac{1}{2}x^2 + \cdots, \quad \omega_{24} = \tfrac{1}{3}x^3 + \cdots, \quad \omega_{34} = \tfrac{1}{12}x^4 + \cdots.$$

Therefore, *the equation of the osculating linear complex, referred to the special tetrahedron of reference, is*

$$\omega_{14} - \omega_{23} = 0. \tag{42}$$

We might have obtained this complex in another way. For, it is clear that the null-system of the osculating cubic will be the same as that determined by the osculating linear complex. We shall, instead, set up the null-system of the torsal cubic in its general form. We shall see that *the linear complex determined by the torsal cubic coincides with the osculating linear complex.*

We have the equations of the torsal cubic

$$\begin{aligned} x_1 &= 60\theta_3 + 6P_2' + 24P_2\eta + 15\eta^3, \\ x_2 &= 24P_2 + 30\eta^2, \qquad x_3 = 30\eta, \qquad x_4 = 20. \end{aligned} \tag{43}$$

The coördinates of the plane, which osculates the torsal cubic at the point whose parameter is η, are

$$\begin{aligned} u_1 &= -180, \qquad u_2 = 270\eta, \qquad u_3 = 144P_2 - 270\eta^2, \\ u_4 &= 540\theta_3 + 54P_2'\eta - 324P_2\eta + 135\eta^3. \end{aligned} \tag{44}$$

If we put in (43) $\eta = \eta_k (k = 1, 2, 3)$ we obtain three points on the cubic. The coördinates of their plane must be proportional to

$$\begin{aligned} v_1 &= -180, \qquad v_2 = 90(\eta_1 + \eta_2 + \eta_3), \\ v_3 &= 144P_2 - 90(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2), \\ v_4 &= 540\theta_3 + 54P_2' - 108P_2(\eta_1 + \eta_2 + \eta_3) + 135\eta_1\eta_2\eta_3, \end{aligned} \tag{45}$$

for each of these expressions must be a symmetric function of η_1, η_2, η_3 of order not higher than the third, and for $\eta_1 = \eta_2 = \eta_3 = \eta$ we must have v_k proportional to u_k. Similarly, the point in which the three osculating planes at η_1, η_2, η_3 intersect, must have its coördinates proportional to

$$\begin{aligned} \omega x_1 &= 60\theta_3 + 6P_2' + 8P_2(\eta_1 + \eta_2 + \eta_3) + 15\eta_1\eta_2\eta_3, \\ \omega x_2 &= 24P_2 + 10(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2), \\ \omega x_3 &= 10(\eta_1 + \eta_2 + \eta_3), \qquad \omega x_4 = 20. \end{aligned} \tag{46}$$

If we eliminate η_1, η_2, η_3 between (45) and (46) and change slightly the factor of proportionality, we find

$$\begin{aligned} \omega v_1 &= -x_4, \qquad \omega v_3 = -x_2 + 2P_2x_4, \\ \omega v_2 &= +x_3, \qquad \omega v_4 = +x_1 - 2P_2x_3, \end{aligned} \tag{47a}$$

or

$$\begin{aligned} \omega' x_1 &= +2P_2v_2 + v_4, \qquad \omega' x_3 = +v_2, \\ \omega' x_2 &= -2P_2v_1 - v_3, \qquad \omega' x_4 = -v_1, \end{aligned} \tag{47b}$$

as the equations of the null-system defined by the torsal cubic.

A point y_1, y_2, y_3, y_4 lies in the plane corresponding to x_1, x_2, x_3, x_4 if

$$\sum_{k=1}^{4} v_k y_k = 0.$$

Therefore, the lines which pass through the point x_1, x_2, x_3, x_4 and lie in the plane corresponding to it in the null-system, satisfy the equation

$$\omega_{14} - 2P_2\omega_{34} - \omega_{23} = 0. \tag{48}$$

If the tetrahedron of reference be so chosen as to make $P_2 = 0$, this equation is identical with (42). *Therefore* (48) *represents the osculating linear complex*

when the tetrahedron of reference is general. The osculating and torsal cubics are curves of this complex.

If P_2 is finite the complex (48) is not special. We see therefore that only those values of x for which $P_2 = \infty$ can give points of the curve at which five consecutive tangents have a straight line intersector.

Let us proceed to deduce the equation of the osculating linear complex belonging to a point of C_y infinitesimally close to P_y. If we change x by an infinitesimal amount δx, we find for the coördinates of the vertices of the new tetrahedron of reference

$$\bar{y} = y + y'\delta x = (1 - p_1\delta x)y + z\delta x,$$
$$\bar{z} = z + z'\delta x = -P_2\delta x \cdot y + (1 - p_1\delta x)z + \rho\delta x,$$
$$\bar{\rho} = \rho + \rho'\delta x = -(P_3 - P_2')\delta x \cdot y - 2P_2\delta x \cdot z + (1 - p_1\delta x)\rho + \sigma\delta x,$$
$$\bar{\sigma} = \sigma + \sigma'\delta x = -(P_4 - P_3')\delta x \cdot y - 3(P_3 - P_2')\delta x \cdot z - 3P_2\delta x \cdot \rho + (1 - p_1\delta x)\sigma.$$

Therefore, if a point has the coördinates $\bar{x}_1, \cdots, \bar{x}_4$ in the new system of coördinates, and $x_1, \cdots, x_4$ in the old, we shall have

$$x_1 = (1 - p_1\delta x)\bar{x}_1 - P_2\delta x \cdot \bar{x}_2 - (P_3 - P_2')\delta x \cdot \bar{x}_3 - (P_4 - P_3')\delta x \cdot \bar{x}_4,$$
. . . . . .

Therefore the infinitesimal changes in the coördinates in the sense new minus old, will be

$$\begin{aligned}
\delta x_1 &= [p_1 x_1 + P_2 x_2 + (P_3 - P_2')x_3 + (P_4 - P_3')x_4]\delta x,\\
\delta x_2 &= [-x_1 + p_1 x_2 + 2P_2 x_3 + 3(P_3 - P_2')x_4]\delta x,\\
\delta x_3 &= (-x_2 + p_1 x_3 + 3P_2 x_4)\delta x,\\
\delta x_4 &= (-x_3 + p_1 x_4)\delta x.
\end{aligned} \tag{49}$$

Referred to the new tetrahedron of reference, the equation of the complex osculating C_y at the point corresponding to $x + \delta x$ will be

$$\bar{\omega}_{14} - 2\bar{P}_2\bar{\omega}_{34} - \bar{\omega}_{23} = 0, \tag{50}$$

where

$$\bar{P}_2 = P_2 + P_2'\delta x, \qquad \bar{\omega}_{ik} = \bar{x}_i\bar{y}_k - \bar{x}_k\bar{y}_i,$$
$$\bar{x}_i = x_i + \delta x_i, \qquad \bar{y}_i = y_i + \delta y_i,$$

if $\bar{x}_i$ and $\bar{y}_i$ denote the coördinates of two points on a line of the complex referred to the new tetrahedron of reference. Making the calculations we find

$$\bar{\omega}_{14} = \omega_{14} + [-\omega_{13} + 2p_1\omega_{14} - P_2\omega_{42} + (P_3 - P_2')\omega_{34}]\delta x,$$
$$\bar{\omega}_{34} = \omega_{34} + (2p_1\omega_{34} + \omega_{42})\delta x,$$
$$\bar{\omega}_{23} = \omega_{23} + [2p_1\omega_{23} - 3P_2\omega_{42} - \omega_{13} - 3(P_3 - P_2')\omega_{34}]\delta x.$$

If we substitute in (50), we find as the *equation of the linear complex osculating C_y at a point infinitesimally close to P_y*,

$$(51) \qquad (\omega_{14} - 2P_2\omega_{34} - \omega_{23})(1 + 2\rho_1\delta x) + 4\theta_3\omega_{34}\delta x = 0.$$

This coincides with the linear complex osculating C_y at P_y, if, and only if, $\theta_3 = 0$.

Therefore, *if the invariant θ_3 vanishes identically, the tangents of the curve C_y belong to a linear complex.* If it does not vanish identically, those values of x for which it does vanish correspond to points of the curve at which the osculating linear complex hyperosculates the curve.

This result, which has been known a long time, is here derived in a novel manner, which has above all the advantages of clearness. It may also be obtained by setting up the linear differential equation of the sixth order satisfied by the six line coördinates

$$\omega_{ik} = y_i y_k' - y_k y_i',$$

of the tangent, and noting that this reduces to the fifth order if, and only if, $\theta_3 = 0$. This is the method of HALPHEN.

A former result may now be stated as follows: *The osculating and torsal cubics of all points of a curve coincide, if, and only if, the curve belongs to a linear complex.*

§ 5. *Geometrical definition of the fundamental tetrahedron of reference.*

We have seen that there exists for every point of the curve C_y a tetrahedron whose vertices P_y, P_z, P_ρ, P_σ are determined by the choice of the independent variables x. In order that we may be able to obtain a clear insight into the geometry of the curve, it is necessary that we may be able to define this tetrahedron by purely geometrical considerations. As a consequence of our preceding results we are now able to do this.

We have already noticed that P_z is a point on the tangent, and by a properly chosen transformation $\xi = \xi(x)$ it may be transformed into any other point of the tangent. When the independent variable has been definitely chosen to be x, we obtain, therefore, a point P_z on the tangent which is not, in general, distinguished by any geometrical property from any other point of the tangent. Its position may serve as a geometric image of the independent variable.

Consider the osculating conic

$$x_4 = 0, \qquad 40x_1x_3 - 32P_2x_3^2 - 15x_2^2 = 0.$$

The polar of any point $(x_1', x_2', x_3', 0)$ of the osculating plane with respect to it, is the straight line

$$x_4 = 0, \qquad 20x_3'x_1 - 15x_2'x_2 + (20x_1' - 32P_2x_3')x_3 = 0.$$

Therefore, the polar of P_z, whose coördinates are $(0, 1, 0, 0)$, is the line $x_2 = 0, x_4 = 0$. In other words:

The line $P_y P_\rho$ is the polar of P_z with respect to the osculating conic.

We shall speak of the curves C_z, C_ρ, C_σ as the derivative curves of C_y with respect to x, of the first, second and third kind respectively. The ruled surfaces which are obtained by joining the points of C_y to the corresponding points of C_z, C_ρ, C_σ shall be called derivative ruled surfaces of the first, second and third kind respectively. Then, the derivative ruled surface of the first kind is unique. It is simply the developable whose cuspidal edge is C_y. Let us consider the derivative ruled surface S of the second kind generated by $P_y P_\rho$. The curve C_y is, of course, an asymptotic curve upon it. This surface is characterized by the equations (20), where p_1 has been assumed equal to zero.

According to the general theory of ruled surfaces,* the asymptotic tangents to S at the points P_y and P_ρ are obtained by joining these points to $2z$ and

$$2\sigma - 4P_2 z + 2P_3 y$$

respectively. Therefore the asymptotic tangent to S at any point $(\alpha_1, 0, \alpha_3, 0)$ of $P_y P_\rho$ joins this point to

$$2P_3\alpha_3 y + (2\alpha_1 - 4P_2\alpha_3)z + 2\alpha_3\sigma .$$

Hence, the equation of the plane tangent to S at $(\alpha_1, 0, \alpha_3, 0)$ is

$$-\alpha_3 x_2 + (\alpha_1 - 2P_2\alpha_3)x_4 = 0 .$$

To the same point of $P_y P_\rho$ there corresponds a plane in the osculating linear complex. According to $(47a)$ this is the plane

$$\alpha_3 x_2 + (\alpha_1 - 2P_2\alpha_3)x_4 = 0 .$$

Therefore, if at any point of the generator of the derived ruled surface of the second kind we construct the tangent plane as well as the plane which corresponds to it in the osculating linear complex, these planes form an involution. The double planes of this involution are the osculating plane $(x_4 = 0)$, and a plane $(x_2 = 0)$ which contains P_σ, the point of the derivative curve of the third kind which corresponds to P_y.

The point which corresponds to this latter plane, is

$$\beta = 2P_2 y + \rho . \tag{52}$$

According to $(47a)$ we have further, corresponding to the point P_z or $(0, 1, 0, 0)$, the plane $x_3 = 0$, which also contains P_σ. The line $P_y P_\sigma$ is now completely determined, as follows:

* WILCZYNSKI, *Covariants*, etc. Transactions of the American Mathematical Society, vol. 3 (1902), p. 434.

The generator of the derived ruled surface of the third kind is the intersection of the following two planes: 1st, the plane corresponding to P_z in the osculating linear complex; 2d, that plane which is tangent to the derived ruled surface of the second kind at the same point which corresponds to it in the osculating linear complex.

It still remains to determine the position of P_ρ and P_σ on the lines $P_y P_\rho$ and $P_y P_\sigma$.

The osculating conic intersects $P_y P_\rho$ in P_y and in P_α where

$$\alpha = 4P_2 y + 5\rho. \tag{53}$$

The cross-ratio of the four points P_y, P_ρ, P_α, P_β, is

$$(\alpha, y, \beta, \rho) = \tfrac{5}{2}.$$

If, upon the generator of the derived ruled surface of the second kind, there be marked its intersections with the osculating conic, and the point P_β whose tangent plane coincides with the plane corresponding to it in the osculating linear complex, the point P_ρ is determined by the condition that the cross-ratio of these four points shall be equal to $\frac{5}{2}$.

If $P_2 = 0$ this definition of P_ρ breaks down. In that case, however, P_α and P_β coincide with P_y. Therefore, *if the derived ruled surface of the third kind is a developable, P_ρ is that point on the generator of the derived surface of the second kind where this generator intersects the osculating conic the second time. At this point the plane, tangent to the ruled surface, and the plane, corresponding to it in the osculating linear complex, coincide.*

If we use the notations of the theory of ruled surfaces,* we find from (20),

$$u_{12} = 4, \qquad u_{21} = 8P'_3 - 4P_4 + 8P_2^2, \qquad u_{11} - u_{22} = 16P_2,$$
$$(u_{11} - u_{22})^2 + 4u_{12}u_{21} = -64(P_4 - 2P'_3 - 6P_2^2).$$

But $u_{11} - u_{22} = 0$ is the condition that C_y and C_ρ shall be harmonically divided by the branches of the flecnode curves of the ruled surface, while $(u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0$ is the condition under which the two branches of the flecnode curve coincide. † Therefore, we obtain the following theorem.

If the derived ruled surface of the third kind is a developable, the intersections of the generator of the derived ruled surface of the second kind with the osculating conic give rise to two curves upon this surface harmonically conjugate with respect to the two branches of its flecnode curve.

If $\theta_4 = 0$, the second intersection of the generator of this surface with the osculating conic is a point of its flecnode curve. Moreover, the two branches of the flecnode curve must then coincide.

* WILCZYNSKI, *Covariants*, etc., loc. cit., p. 445. *Invariants*, etc., Transactions of the American Mathematical Society, vol. 2 (1901), p. 6.

† Loc. cit., p. 444.

It is to be noted that we have here a *geometrical interpretation for the invariant equation* $\theta_4 = 0$. We shall find two other, quite different interpretations for this condition later on. By the method of HALPHEN still another meaning can be attached to this equation, not however so purely geometrical.

We may, if we wish, make use of the torsal cubic in our further constructions. For, it is now defined entirely by geometrical considerations. If, in fact, we trace upon the developable, whose edge of regression is C_y an arbitrary curve C_z, we now know how the corresponding ruled surfaces of the second and third kind may be constructed. They depend upon an arbitrary function of x, as does the curve C_z. Among the surfaces of the third kind there exists a single one-parameter family of developables. Upon that generator of each of these developables which passes through P_y we mark the point where it intersects the cuspidal edge of the developable to which it belongs. The locus of these points is the torsal cubic.

We notice incidentally that *the reduction of equation* (1) *to the Laguerre-Forsyth canonical form is equivalent to the determination of one of the developables of the third kind.* Since this reduction is made by solving an equation of the RICCATI form we notice further the following theorem. *The four curves on the developable of* C_y *which correspond to any four of the developables of the third kind, intersect all of the tangents of* C_y *in point-rows of the same cross-ratio.*

Let us consider the developable surface of the torsal cubic, which is given by equations (44). We are going to find its intersection with the plane $P_y P_z P_\sigma$, or $x_3 = 0$. The intersection of the plane $u_1, \cdots, u_4$, which osculates the cubic at the point whose parameter is η, with the plane $x_3 = 0$ is the line

$$-180x_1 + 270\eta x_2 + (540\theta_3 + 54P_2' - 324P_2\eta + 135\eta^3)x_4 = 0$$

of this plane. As η changes this line envelops a curve, the required intersection. Its equation will be found by eliminating η between the above equation and this other one

$$270x_2 + (-324P_2 + 405\eta^2)x_4 = 0$$

obtained from it by differentiation with respect to η. This elimination may be easily performed. The result is

$$(54)\quad F = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 + 15x_4\{10x_1 - (30\theta_3 + 3P_2')x_4\}^2 = 0.$$

This plane cubic together with the tangent $P_y P_z$ gives the complete intersection of the plane $P_y P_z P_\sigma$ with the developable of the torsal cubic. It has a cusp at P_z, and the equation of its cusp tangent is

$$5x_1 - 6x_4 = 0,$$

as one may find by the general theory of plane curves. It intersects $P_y P_\sigma$ in the point

$$\kappa = 24y + 20\sigma. \tag{55}$$

The tangent to the plane cubic at P_y is

$$10x_2 + 3x_4 = 0.$$

It intersects the cubic again in the point

$$(12 + 30\theta_3 + 3P_2')y - 6z + 20\sigma.$$

If this point be joined to P_z by a straight line, the latter will intersect $P_y P_\sigma$ in the point

$$\lambda = (12 + 30\theta_3 + 3P_2')y + 20\sigma. \tag{56}$$

The plane $P_y P_z P_\sigma$ is tangent to the torsal cubic. It intersects it once more in the point corresponding to $\eta = 0$, viz., $(60\theta_3 + 6P_2')y + 24P_2 z + 20\sigma$. A line joining this point to P_z intersects $P_y P_\sigma$ in

$$\mu = (60\theta_3 + 6P_2')y + 20\sigma. \tag{57}$$

Consider the four points P_κ, P_λ, P_μ and P_y. We have

$$\lambda = \frac{\mu + \kappa}{2}, \qquad (30\theta_3 + 3P_2' - 12)y = \frac{\mu - \kappa}{2};$$

so that P_λ is the harmonic conjugate of P_y with respect to P_κ and P_μ.

The osculating cubic differs from the torsal cubic only in having $2(P_3 - P_2')$ in place of $10\theta_3 + P_2'$. Consequently the plane cubic in which its developable intersects the plane $P_y P_z P_\sigma$ is

$$\bar{F} = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 + 15x_4\{10x_1 - 6(P_3 - P_2')x_4\}^2 = 0. \tag{58}$$

If we denote by $P_{\bar\kappa}$, $P_{\bar\lambda}$, $P_{\bar\mu}$ the points constructed with respect to this curve in the same way as P_κ, and P_λ and P_μ were with respect to $F = 0$, we find

$$\bar\kappa = \kappa, \quad \bar\lambda = (12 + 6P_3 - 6P_2')y + 20\sigma, \quad \bar\mu = (12P_3 - 12P_2')y + 20\sigma, \tag{59}$$

the cusp and its tangent being common to the two curves, as well as the tangent at P_y. I refrain from formulating explicitly the various theorems which may be obtained from these equations.

In order to obtain a simple construction for P_σ, we shall consider finally the developable generated by the motion of the plane $P_y P_\rho P_\sigma$. The equation of this plane is $x_2 = 0$. As x changes into $x + \delta x$, y, z, ρ change into $y + y'\delta x$, $\rho + \rho'\delta x$, $\sigma + \sigma'\delta x$ respectively, where y', ρ', σ' are given by equations (5). The equation of the plane of these points, referred to the tetrahedron $P_y, P_z, P_\rho, P_\sigma$, is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1-p_1\delta x & \delta x & 0 & 0 \\ -(P_3-P_2')\delta x & -2P_2\delta x & 1-p_1\delta x & \delta x \\ -(P_4-P_3')\delta x & -3(P_3-P_2')\delta x & -3P_2\delta x & 1-p_1\delta x \end{vmatrix} = 0,$$

which becomes, when developed,

$$x_1\delta x - x_2(1-3p_1\delta x) - 2x_3P_2\delta x - 3x_4(P_3-P_2')\delta x = 0.$$

Therefore the equations of the generator of the developable, generated by the motion of the plane $P_\gamma P_\rho P_\sigma$, are

$$x_2 = 0, \qquad x_1 - 2P_2x_3 - 3(P_3-P_2')x_4 = 0. \tag{60}$$

It intersects the generator $P_\gamma P_\rho$ ($x_2 = x_4 = 0$) of the derived ruled surface of the second kind in that point

$$\beta = 2P_2y + \rho \tag{52}$$

whose tangent plane coincides with the plane corresponding to it in the osculating linear complex. Its intersection with $P_\gamma P_\sigma$, the generator of the derived ruled surface of the third kind, is

$$\gamma = 3(P_3-P_2')y + \sigma. \tag{61}$$

The generator of the developable joins P_β to P_γ. We wish to determine its edge of regression. If

$$\delta = l\beta + m\gamma$$

is the point where $P_\beta P_\gamma$ meets the edge of regression, we must have

$$\delta' = r\beta + s\gamma, \qquad \text{or} \qquad l\beta' + m\gamma' = \bar{r}\beta + \bar{s}\gamma.$$

We proceed to determine the ratio of l to m. We find

$$\beta' = (3P_2' - 2p_1P_2 - P_3)y - p_1\rho + \sigma,$$

$$\gamma' = -[P_4 - 4P_3' + 3P_2'' + 3p_1(P_3-P_2')]y - 3P_2\rho - p_1\sigma.$$

We may eliminate ρ and σ by (52) and (61). This gives

$$\beta' = -4\theta_3y - p_1\beta + \gamma,$$

$$\gamma' = -(P_4 - 4P_3' + 3P_2'' - 6P_2^2)y - 3P_2\beta - p_1\gamma.$$

We may therefore put

$$l = P_4 - 4P_3' + 3P_2'' - 6P_2^2, \qquad m = -4\theta_3;$$

so that

$$\delta = (P_4 - 4P_3' + 3P_2'' - 6P_2^2)\beta - 4\theta_3\gamma \tag{62}$$

gives the edge of regression. This gives the following theorem:

The developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, has its edge of regression upon the derived ruled surface of the second kind, if, and only if, the curve C_y belongs to a linear complex.

We may write in place of (51)

$$\gamma = 60(P_3 - P_2')y + 20\sigma .$$

We have from (59)

$$\bar{\mu} = 12(P_3 - P_2')y + 20\sigma .$$

Therefore, the cross-ratio of the four points P_γ, P_y, $P_{\bar{\mu}}$ and P_σ is

$$(\gamma, y, \bar{\mu}, \sigma) = 5 .$$

We have found finally a geometrical definition for P_σ, which we may recapitulate as follows. *The plane of the tangent and the generator of the third derived ruled surface intersects the osculating cubic in P_y counted twice and one other point. If the latter point be joined to P_z by a straight line we obtain a certain point $P_{\bar{\mu}}$ as the intersection of this line with $P_y P_\sigma$. The generator of the developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, intersects $P_y P_\sigma$ in another point P_γ. P_σ may now be found as that point of $P_y P_\sigma$ which makes the cross-ratio*

$$(P_\gamma, P_y, P_{\bar{\mu}}, P_\sigma) = 5 .$$

We have shown how to construct the fundamental tetrahedron when P_z is given. If P_ρ is given, P_z can be found at once as the pole of $P_y P_\rho$ with respect to the osculating conic. If P_σ is given we may find first its polar plane with respect to the osculating linear complex which is

$$-x_1 + 2P_2 x_3 = 0 ,$$

and therefore passes through P_z, but not through P_y. P_z can therefore be found at once at the intersection of this plane with the tangent to C_y at P_y.

We see therefore, that *any one of the three points P_z, P_ρ, P_σ determines uniquely the others.*

§ 6. *Some further properties of the derived ruled surfaces of the second and third kind.*

Let us suppose $p_1 = P_2 = 0$, so that the derived ruled surface of the third kind is a developable, and let us consider the derived ruled surface of the second kind which corresponds to it. We proceed to deduce the equation of its osculating linear complex.

Let Y and R denote the developments of y and ρ in the vicinity of the ordi-

nary point $x = a$, and replace again $x - a$ by x in the developments. Then we shall find

$$Y = y_1 y + y_2 z + y_3 \rho + y_4 \sigma, \qquad R = \rho_1 y + \rho_2 z + \rho_3 \rho + \rho_4 \sigma,$$

where

$$\begin{aligned}
\rho_1 &= -P_3 x - \tfrac{1}{2} P_4 x^2 - \tfrac{1}{6} P_4' x^3 + \tfrac{1}{24}(4P_3^2 - P_4'') x^4 + \cdots, \\
\rho_2 &= -2P_3 x^2 - \tfrac{1}{6}(4P_3' + P_4) x^3 - \tfrac{1}{12}(2P_3'' + P_4') x^4 + \cdots, \\
\rho_3 &= 1 - \tfrac{2}{3} P_3 x^3 - \tfrac{1}{24}(8P_3' + P_4) x^4 + \cdots, \\
\rho_4 &= x - \tfrac{1}{6} P_3 x^4 + \cdots,
\end{aligned} \tag{63}$$

while $y_1, \cdots, y_4$ have been computed before. Denote by

$$\omega_{ik} = y_i \rho_k - y_k \rho_i$$

the Plückerian coördinates of the line joining the two points. We find

$$\begin{aligned}
\omega_{12} &= -P_3 x^2 - \tfrac{1}{3}(2P_3' - P_4) x^3 - \tfrac{1}{12}(2P_3'' - P_4') x^4 + \cdots, \\
\omega_{13} &= 1 - \tfrac{1}{3} P_3 x^3 - \tfrac{1}{6}(2P_3' - P_4) x^4 + \cdots, \\
\omega_{14} &= x - \tfrac{1}{6} P_3 x^4 + \cdots, \\
\omega_{23} &= x + \tfrac{1}{6} P_3 x^4 + \cdots, \\
\omega_{24} &= x^2 + \cdots, \qquad \omega_{34} = \tfrac{1}{3} x^3 + \cdots.
\end{aligned}$$

Let

$$a\omega_{12} + b\omega_{12} + c\omega_{14} + d\omega_{23} + e\omega_{24} + f\omega_{34} = 0$$

be the equation of the osculating linear complex of the surface in question. Then, the coefficients of all powers of x up to and including x^4 must be zero, if we substitute the above developments of ω_{ik} into the left member. This gives us the following equations:

$$\begin{gathered}
b = 0, \qquad c + d = 0, \qquad -P_3 a + e = 0, \\
-\tfrac{1}{3}(2P_3' - P_4) a - \tfrac{1}{3} P_3 b + \tfrac{1}{3} f = 0, \\
-\tfrac{1}{12}(2P_3'' - P_4') a - \tfrac{1}{6}(2P_3' - P_4) b - \tfrac{1}{6} P_3 c + \tfrac{1}{6} P_3 d = 0,
\end{gathered}$$

whence the ratios of the coefficients may be easily deduced.

We find thus the equation of the linear complex osculating the derived ruled surface of the second kind which corresponds to a developable of the third kind; it is

$$-4P_3\omega_{12} - (P_4' - 2P_3'')(\omega_{14} - \omega_{23}) - 4P_3^2\omega_{24} + 4(P_4 - 2P_3')P_3\omega_{34} = 0. \tag{64}$$

It coincides with the osculating linear complex of C_y if and only if $P_3 = 0$, i. e., if C_y belongs to a linear complex. This result is also obvious for geometrical reasons.

The coördinates v_k of the plane, which corresponds to a point x_1, x_2, x_3, x_4 in the linear complex (64), are given by

$$
\begin{aligned}
\omega v_1 &= 4P_3 x_2 + (P_4' - 2P_3'')x_4, \\
\omega v_2 &= -4P_3 x_1 - (P_4' - 2P_3'')x_3 + 4P_3^2 x_4, \\
\omega v_3 &= (P_4' - 2P_3'')x_2 - 4P_3(P_4 - 2P_3')x_4, \\
\omega v_4 &= -(P_4' - 2P_3'')x_1 - 4P_3^2 x_2 + 4P_3(P_4 - 2P_3')x_3,
\end{aligned} \tag{65}
$$

where ω is a proportionality factor.

Let us consider at the same time the osculating linear complex of C_y. The lines common to the two complexes form a congruence whose directrices we propose to find. This we can do quite easily by writing down the equations which express that, for a point on one of the directrices, the two planes corresponding to it in the two complexes must coincide. The right members of (65) must, for such a point be equal to

$$-\omega x_4, \qquad +\omega x_3, \qquad -\omega x_2, \qquad +\omega x_1$$

respectively, where ω is a proportionality factor.

The four equations obtained in this way can be satisfied only if their determinant vanishes, which gives

$$2P_3'' - P_4' - \omega = \pm 4P_3 \sqrt{2P_3' - P_4}, \tag{66}$$

whence the following equations for the two directrices

$$
\begin{aligned}
\pm \sqrt{2P_3' - P_4}\,x_1 - P_3 x_2 - (2P_3' - P_4)x_3 &= 0, \\
-x_2 \pm \sqrt{2P_3' - P_4}\,x_4 &= 0, \\
-x_1 \pm \sqrt{2P_3' - P_4}\,x_3 + P_3 x_4 &= 0,
\end{aligned} \tag{67}
$$

of which three equations only two are independent, and where we have assumed $P_3 \neq 0$. In fact, if P_3 were zero the two complexes would coincide and the congruence would be indeterminate.

Since we have assumed $P_2 = 0$, the quantity under the square root is $-\theta_4$. We find a second interpretation for the condition $\theta_4 = 0$. *If $\theta_4 = 0$, the congruence has coincident directrices.* We may combine this with our former result to the following theorem.

Choose as derived ruled surface of the third kind one of the developables of the single family which exists. Consider the osculating linear complex of the corresponding ruled surface S of the second kind. Let the directrices of the congruence, which this complex has in common with the osculating linear complex of the fundamental curve C_y, coincide. Then the two branches of the

flecnode curve of S coincide and the generator of S which passes through P_y will intersect the osculating conic of C_y in P_y, and a second point whose locus is the flecnode curve of the surface S.

I refrain from formulating the converse. The above conditions are fulfilled if and only if $\theta_4 = 0$.

Let us consider one of the directrices (67), for example that one which corresponds to the plus sign of the square root. Then we see that

$$\begin{aligned} \alpha &= P_3\sqrt{2P_3' - P_4}\,y + P_3\rho, \\ \beta &= \sqrt{2P_3' - P_4}\,(P_3 y + \sqrt{2P_3' - P_4}\,z + \sigma) \end{aligned} \tag{68}$$

are two points on the directrix. We have multiplied each expression by a factor so as to have α and β of the same weight. If now we change the independent variable, but in such a way as not to disturb the condition $P_2 = 0$, we shall get in (64) a single infinity of complexes, and in (67) two families of lines, the directrices of the single infinity of congruences which thus result. We are going to study to some extent the two ruled surfaces thus generated.

Put

$$k = \sqrt{2P_3' - P_4}. \tag{69}$$

Making the transformations which preserve $P_2 = 0$, for which we must have

$$\mu = \eta' - \tfrac{1}{2}\eta^2 = 0,$$

we find that α and β are transformed into $\bar{\alpha}$ and $\bar{\beta}$, where

$$\begin{aligned} (\xi')^5\bar{\alpha} &= \alpha + 2\eta P_3 z + \tfrac{3}{2}\eta^2 P_3 y, \\ (\xi')^5\bar{\beta} &= \beta + \tfrac{3}{2}\overset{k}{P_3}\eta\alpha + k(\tfrac{3}{2}\eta^2 z + \tfrac{3}{4}\eta^3 y). \end{aligned}$$

The point $m\bar{\alpha} + n\bar{\beta}$ will be an arbitrary point on the line joining $P_{\bar{\alpha}}P_{\bar{\beta}}$. We find therefore, the equations of our surface referred to two parameters η and m/n,

$$\begin{aligned} x_1 &= \tfrac{3}{2}mP_3\eta^2 + (m+n)P_3 k + n(\tfrac{3}{2}\eta k^2 + \tfrac{3}{4}\eta^3 k), \\ x_2 &= 2m\eta P_3 + n(k^2 + \tfrac{3}{2}\eta^2 k), \\ x_3 &= mP_3 + \tfrac{3}{2}kn\eta, \\ x_4 &= nk. \end{aligned} \tag{70}$$

The nature of this surface may be easily determined. We have, returning to the two curves C_α and C_β upon it, for C_α

$$\alpha_1 = k + \tfrac{3}{2}\eta^2, \qquad \alpha_2 = 2\eta, \qquad \alpha_3 = 1, \qquad \alpha_4 = 0,$$

if we put $n = 0$, $m = 1/P_3$ in (70). If we put $m = 0$, $n = 1/k$ we find for C_β

$$\beta_1 = P_3 + \tfrac{3}{2}k\eta + \tfrac{3}{4}\eta^3, \qquad \beta_2 = k + \tfrac{3}{2}\eta^2, \qquad \beta_3 = \tfrac{3}{2}\eta, \qquad \beta_4 = 1.$$

But (α_k, β_k) are simultaneous solutions of the equation

$$\frac{d\beta}{d\eta} = \tfrac{3}{2}\alpha,$$

which proves that *the ruled surface which we are considering is a developable whose edge of regression is the twisted cubic* C_β.

The curve C_α is a conic, the intersection of the developable of the cubic with the osculating plane of C_y. Its equations are

$$x_4 = 0, \qquad -3x_2^2 + 8x_1x_3 - 8kx_3^2 = 0.$$

We notice that for $\theta_4 = 0$ *it coincides with the osculating conic*, a further interpretation of this condition. *In general, the two conics have a contact of third order at* P_y.

If, in these equations, we change k into $-k$ we obtain the developable, cubic and conic associated with the second directrix of our congruence. A considerable number of other configurations are suggested by the combinations of these various curves and surfaces. I will refrain, however, from any further study in this direction.

The curve C_y is an asymptotic curve upon every derived ruled surface of the second kind. Moreover, the most general derived ruled surface of the second kind depends upon one arbitrary function, as does also the most general ruled surface containing C_y as an asymptotic curve. It is easy to see that the derived ruled surface of the second kind may be made to coincide with any ruled surface upon which C_y is an asymptotic curve if the independent variable be properly chosen.

Upon the derived ruled surface of the third kind, C_y can never be an asymptotic curve. It may, however, be one branch of the flecnode curve. In fact, if we form the quantities u_{ik} of the theory of ruled surfaces for system (27), we find

$$u_{12} = -\frac{P_2'}{P_2^2}.$$

But $u_{12} = 0$ is the condition that C_y may be a branch of the flecnode curve on the surface generated by $P_y P_\sigma$. Suppose that the variable has been so chosen as to make $P_2' = 0$. The most general transformation which is possible, leaving this relation invariant, must, according to (11) satisfy the condition

$$-2\eta P_2 - \tfrac{5}{6}\mu' + \tfrac{5}{3}\eta\mu = 0,$$

or

$$-5\eta'' - 12\eta P_2 + 15\eta\eta' - 5\eta^3 = 0, \tag{71}$$

a differential equation of the second order for η. Moreover two different solutions of this equation always give rise to two distinct ruled surfaces. For, let

η_1 and η_2 be two such solutions, and let σ_1, σ_2 be the corresponding values of σ. Then, according to (9),

$$\sigma_\kappa = \frac{1}{(\xi'_\kappa)^3}\left[\sigma + \tfrac{3}{2}\eta_\kappa\rho + (\mu_\kappa + \tfrac{3}{2}\eta_\kappa^2)z + \tfrac{1}{4}(\mu'_\kappa + 4\eta_\kappa\mu_\kappa + 3\eta_\kappa^3)y\right] \quad (\kappa = 1, 2),$$

But if the same ruled surface corresponds to η_1 and η_2, the three points y, σ_1 and σ_2 must be collinear. We must therefore be able to reduce

$$(\xi'_1)^3\sigma_1 - (\xi'_2)^3\sigma_2 = \tfrac{3}{2}(\eta_1 - \eta_2)\rho + \cdots$$

to a multiple of y. But this is clearly only possible if $\eta_2 = \eta_1$.

We see, therefore, that *there are ∞^2 derived ruled surfaces of the third kind upon which C_y is one branch of the flecnode curve.*

If P'_2 is not zero, our problem leads to the differential equation

$$\frac{d}{dx}(\eta' - \tfrac{1}{2}\eta^2) = \tfrac{6}{5}P'_2 - \tfrac{1}{5}^2 P_2\eta + 2\eta(\eta' - \tfrac{1}{2}\eta^2), \tag{72}$$

which is of the second order and third degree.

I have shown in a former paper that the most general ruled surface which has C_y as one branch of its flecnode curve contains an arbitrary function in its general expression.* I have also shown that together with any such surface, its flecnode surface and each member of a single infinity of surfaces determined by these two, also contains C_y as one branch of its flecnode curve. One might imagine that there could be based upon these theorems a transformation theory of equation (72). This is not the case however. For, if one of the surfaces containing C_y as a branch of its flecnode curve is a derived ruled surface of the third kind, its flecnode surface is not, nor is any member of the family of ruled surfaces just mentioned.

Corresponding to the ∞^2 solutions of (72) or of

$$\mu' = \tfrac{6}{5}P'_2 - \tfrac{1}{5}^2 P_2\eta + 2\eta\mu,$$

we find ∞^2 positions for $P_{\bar{\sigma}}$, viz.:

$$4(\xi')^3\bar{\sigma} = (\tfrac{6}{5}P'_2 - \tfrac{1}{5}^2 P_2\eta + 6\eta\mu + 3\eta^3)y + (4\mu + 6\eta^2)z + 6\eta\rho + 4\sigma.$$

The locus of these points is a cubic surface

$$27P'_2x_4^3 - 36P_2x_3x_4^2 + 90x_2x_3x_4 - 90x_1x_4^2 - 40x_3^3 = 0, \tag{73}$$

which contains P_yP_z, the tangent of C_y as a double line. It is therefore a ruled surface. It is in fact a CAYLEY's cubic scroll.

If one derived surface of the third kind is known upon which C_y is a branch

* *General theory of curves on ruled surfaces*, Transactions of the American Mathematical Society, vol. 6 (1905), p. 78.

of the flecnode curve, two others may be found by merely solving a quadratic equation.

In fact, suppose that a solution η of (72) be known. We may make a transformation of the independent variables, $\xi = \xi(x)$ such that

$$\frac{\xi''}{\xi'} = \eta.$$

In the resulting equation $P_2' = 0$. If we again denote the independent variable by x, (72) becomes

$$\eta'' - \eta\eta' = -\tfrac{1}{5}^2 P_2 \eta + 2\eta(\eta' - \tfrac{1}{2}\eta^2),$$

where P_2 is a constant, since $P_2' = 0$. But we may satisfy this equation by putting $\eta = \text{const.}$, which gives the equation

$$\eta^3 + \tfrac{1}{5}^2 P_2 \eta = 0,$$

whence

$$\eta = 0, \qquad \pm\sqrt{-\tfrac{1}{5}^2 P_2}.$$

The root $\eta = 0$ gives the original solution. The other two are new.

§7. *The principal tangent plane of two space curves. The covariants. Transition to Halphen's investigations.*

HALPHEN has introduced a very important notion, which we shall now proceed to explain.

Let there be given two space curves having at a point P a contact of the nth order. If these curves be projected from any center Q upon a plane, the projections will also have, in general, a contact of the nth order at the point corresponding to P. HALPHEN shows that there exists a plane passing through the common tangent of the two curves such that if the center of projection be taken anywhere within it, the contact of the projections will be of an order higher than n. This plane he calls the *principal tangent plane* of the two curves.*

We shall follow HALPHEN in determining the principal tangent plane at P_y of the curve C_y and its osculating cubic. This will lead us to an especially simple form for the development of the equations of the curve, which is also due to HALPHEN and on the basis of which he draws his further conclusions. It will also enable us to substitute for our system of covariants C_2, C_3, C_4 another system whose geometrical significance will be apparent, and in terms of which C_2, C_3, C_4 may be expressed.

* HALPHEN, *Sur les invariants différentiels des courbes gauches*, Journal de l'Ecole Polytechnique, vol. 47 (1880), p. 25.

Assuming $P_2 = 0$, the equations of the osculating cubic, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, are

$$3x_2x_4 - 2x_3^2 = 0, \qquad 2x_3(5x_1 - 3P_3x_4) - 5x_2^2 = 0.$$

Let us put

$$\bar{x}_1 = x_2, \qquad \bar{x}_2 = \tfrac{2}{5}x_3, \qquad \bar{x}_3 = \tfrac{6}{25}x_4, \qquad \bar{x}_4 = 5x_1 - 3P_3x_4, \tag{74}$$

and

$$x = \frac{\bar{x}_1}{\bar{x}_4}, \qquad y = \frac{\bar{x}_2}{\bar{x}_4}, \qquad z = \frac{\bar{x}_3}{\bar{x}_4}, \tag{75}$$

Then the equations of the cubic reduce to

$$y = x^2, \qquad z = x^3. \tag{76}$$

The relation of the new tetrahedron of reference to the cubic is quite simple. The plane $\bar{x}_3 = 0$ is the osculating plane at P; $\bar{x}_2 = 0$ is some other plane through the tangent; this plane intersects the cubic in another point Q; the plane tangent to the cubic at Q and passing through P is $\bar{x}_1 = 0$; the osculating plane at Q is $\bar{x}_4 = 0$.

Since $\bar{x}_2 = 0$ may be chosen in an infinity of ways we see that the reduction of the equations of a space cubic to the form (76) may be accomplished in an infinity of ways.

For the curve C_y we have

$$\begin{aligned}
y_1 &= 1 - \frac{P_3}{3!}x^3 - \frac{P_4}{4!}x^4 - \frac{P_4'}{5!}x^5 - \frac{P_4'' - 4P_3^2}{6!}x^6 \\
&\qquad - \frac{P_4^{(3)} - 5P_3P_4 - 12P_3P_3'}{7!}x^7 + \cdots, \\
y_2 &= x - \frac{P_3}{3!}x^4 - \frac{P_4 + 4P_3'}{5!}x^5 - \frac{2P_4' + 4P_3''}{6!}x^6 \\
&\qquad + \frac{16P_3^2 - 3P_4'' - 4P_3^{(3)}}{7!}x^7 + \cdots, \\
y_3 &= \tfrac{1}{2}x^2 - \frac{4P_3}{5!}x^5 - \frac{1}{6!}(P_4 + 8P_3')x^6 - \frac{1}{7!}(3P_4' + 12P_3'')x^7 + \cdots, \\
y_4 &= \tfrac{1}{6}x^3 - \frac{4}{6!}P_3x^6 - \frac{1}{7!}(P_4 + 12P_3')x^7 \cdots.
\end{aligned} \tag{77}$$

If we refer it to the same tetrahedron to which we have just referred the osculating cubic, and if we put

$$\xi = \frac{\bar{y}_1}{\bar{y}_4}, \qquad \eta = \frac{\bar{y}_2}{\bar{y}_4}, \qquad \zeta = \frac{\bar{y}_3}{\bar{y}_4},$$

we shall find

$$\eta = \xi^2 + \frac{125}{36}(8P_3' - 5P_4)\xi^6 + \frac{625}{7.36}(8P_3'' - 5P_4')\xi^7 + \cdots,$$

(78)

$$\zeta = \xi^3 - \frac{25}{3}P_3\xi^6 + \frac{125}{7.12}(36P_3' - 25P_4)\xi^7 + \cdots.$$

If we put

(79) $$\xi = \epsilon X, \qquad \eta = \epsilon^2 Y, \qquad \zeta = \epsilon^3 Z, \qquad \epsilon^3 = -\frac{3}{25P_3},$$

assuming, therefore, that P_3 is not zero, this becomes

(80)

$$Y = X^2 + \lambda_6 X^6 + \lambda_7 X^7 + \cdots,$$

$$Z = X^3 + \mu_6 X^6 + \mu_7 X^7 + \cdots,$$

where

(81)

$$\lambda_6 = \tfrac{125}{36}(8P_3' - 5P_4)\epsilon^4, \qquad \lambda_7 = \frac{625}{7.36}(8P_3'' - 5P_4')\epsilon^5,$$

$$\mu_6 = 1, \qquad \mu_7 = \frac{125}{7.12}(36P_3' - 25P_4)\epsilon^4.$$

But we can make a further change of coördinates without disturbing the form (80) of the development. For the plane $\bar{x}_2 = 0$ may be taken to be any plane through the tangent. It may be taken in such a way as to make λ_6 vanish, as we shall presently see. The plane thus obtained will obviously be the principal tangent plane of the curve C_y and its osculating cubic.

Instead of working out the transformation geometrically we shall put with HALPHEN

(82a) $$\bar{X} = \frac{\xi_1}{\omega_1}, \qquad \bar{Y} = \frac{\eta_1}{\omega_1}, \qquad \bar{Z} = \frac{Z_1}{\omega_1},$$

where

(82b)

$$\omega_1 = 1 + 3pX + 3p^2Y + p^3Z, \qquad \eta_1 = Y + pZ,$$

$$\xi_1 = X + 2pY + p^2Z,$$

the quantity p being arbitrary. We shall find with him

(83)

$$\bar{Y} = \bar{X}^2 + \Lambda_6\bar{X}^6 + \Lambda_7\bar{X}^7 + \cdots,$$

$$\bar{Z} = \bar{X}^3 + M_6\bar{X}^6 + M_7\bar{X}^7 + \cdots,$$

as the result of applying this transformation to any system of equations of the form (80). The coefficients of (83) are

(84)

$$M_6 = \mu_6, \qquad M_7 = \mu_7 + 3p\mu_6,$$

$$\Lambda_6 = \lambda_6 + p\mu_6, \qquad \Lambda_7 = \lambda_7 + p\mu_7 + p(p\mu_7 - \lambda_6).$$

For

$$p = -\frac{\lambda_6}{\mu_6},$$

this reduces to the canonical form required, in which we may also assume $\mu_6 = 1$ if the curve does not belong to a linear complex.

If we combine the various transformations which we have made successively we get the following result: *If we introduce non-homogeneous coördinates by putting*

$$X = \frac{\bar{y}_1}{\bar{y}_4}, \qquad Y = \frac{\bar{y}_2}{\bar{y}_4}, \qquad Z = \frac{\bar{y}_3}{\bar{y}_4}, \tag{85a}$$

where

$$\begin{aligned} \bar{y}_1 &= 25\epsilon^2 y_2 + 20p\epsilon y_3 + 6p^2 y_4, \\ \bar{y}_2 &= 10\epsilon y_3 + 6p y_4, \\ \bar{y}_3 &= 6y_4, \\ \bar{y}_4 &= 125\epsilon^3 y_1 + 75p\epsilon^2 y_2 + 30p^2\epsilon y_3 + (6p^3 - 75P_3\epsilon^3)y_4, \end{aligned} \tag{85b}$$

and where

$$\epsilon^3 = -\frac{3}{25P_3}, \qquad p = -\tfrac{1}{3}\tfrac{2}{6}\tfrac{5}{}(8P_3' - 5P_4)\epsilon^4, \tag{85c}$$

the development of the equations of the curve C_y may be written in the canonical form

$$\begin{aligned} Y &= X^2 + \Lambda_7 X^7 + \cdots, \\ Z &= X^3 + X^6 + M_7 X^7 + \cdots, \end{aligned} \tag{86}$$

where Λ_7, M_7, $\cdots$ are given by (84) *together with* (81). *This transformation is valid if C_y does not belong to a linear complex.*

The coefficients of (86) are absolute invariants. We find, in fact,

$$\Lambda_7 = \frac{625\epsilon^5}{7.12\cdot 36\theta_3}(25\theta_4^2 + 20\theta_8 - 4\theta_{3,1}), \qquad M_7 = \frac{625\epsilon^4}{42}\theta_4, \tag{87}$$

where

$$\theta_8 = 4\theta_4\theta_3' - 3\theta_3\theta_4'. \tag{88}$$

From (86) and (76) it is clear that the plane $Y = 0$ or $\bar{y}_2 = 0$ is the principal plane of the curve C_y and its osculating cubic. In the original system of coördinates its equation will, therefore, be

$$4P_3 x_3 + (8P_3' - 5P_4)x_4 = 0. \tag{89}$$

If C_y belongs to a linear complex, it coincides with the osculating plane.

The point which corresponds to (89) in the osculating linear complex is

$$(5P_4 - 8P_3')y + 4P_3 z, \tag{90}$$

or in invariant form

$$(\theta_3' + \tfrac{5}{2}\theta_4)y + 2\theta_3 z, \tag{91}$$

an expression whose covariance may be verified directly. We shall call this point the *principal point of the tangent.* We thus obtain a curve on the developable of C_y which may be called *its principal curve.*

If we make a transformation such that

$$3\eta = \theta_3' + \tfrac{5}{2}\theta_4, \tag{92}$$

the point P_z describes the principal curve on the developable of C_y. The points P_ρ and P_σ also describe perfectly definite curves, whose expressions may be obtained from (9) by substituting for η the expression (92). *Any covariant may be expressed in terms of y and the three which we have just determined, which may, therefore, serve to replace the covariants C_2, C_3 and C_4.*

If C_y belongs to a linear complex these four curves all coincide, so that a different set of fundamental covariants must then be selected.

In this exceptional case our fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ gives rise to a most remarkable configuration. If we put again $P_2 = 0$, we have in this case also $P_3 = 0$. $P_y P_\sigma$ generates one of the developables of the third kind of which C_σ is the cuspidal edge, while $P_y P_z$ of course generates the developable of which C_y is the cuspidal edge. The surface generated by $P_y P_\rho$ is not developable. Its equations become

$$\rho'' + P_4 y = 0, \qquad y'' - \rho = 0.$$

It belongs to the same linear complex as C_y, and C_y and C_ρ are the two branches of its complex curve which is at the same time an asymptotic curve. $P_z P_\rho$ generates a developable, since $z' = \rho$, of which C_z is the cuspidal edge. $P_\rho P_\sigma$ generates a developable, since $\rho' = \sigma$, of which C_ρ is the cuspidal edge. Finally $P_z P_\sigma$ generates a ruled surface whose equations are

$$\sigma'' - \frac{P_4'}{P_4}\sigma' + P_4 z = 0, \qquad z'' - \sigma = 0,$$

upon which C_z and C_σ are asymptotic lines.

We find, therefore, the following theorem.

If the curve belongs to a linear complex, we may, in an infinity of ways, choose the fundamental tetrahedron so that four of its edges give rise to developables whose cuspidal edges are described by the four vertices. The other two edges of the tetrahedron will then give rise to ruled surfaces upon each of which the vertices of the tetrahedron trace a pair of asymptotic curves. The latter coincide with the two branches of the complex curve for the derived surface of the second kind.

It does not seem convenient to develop the geometrical theory of the covariants for this case at the present moment, as it will appear naturally in connec-

tion with some further investigations which I have not yet completed, and which I hope to present on some future occasion.

Those results which HALPHEN has deduced for the general case, and which have not been mentioned in this paper might now be directly transcribed from his memoir. For we have connected our own theory with his by obtaining the canonical form (86) for the development of the equations of the curve. In the case $\theta_3 = 0$, a different canonical form must be used, which has also been indicated by HALPHEN, but not extensively employed. The discussion of this we shall leave for a later occasion together with the investigations mentioned above.

CAMBRIDGE, ENGLAND,
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SUR LES OPÉRATIONS LINÉAIRES*

(DEUXIÈME NOTE†)

PAR

MAURICE FRÉCHET

Nouvelles représentations d'une opération linéaire.

Remarque sur le théorème de M. Hadamard. D'après ce théorème, toute opération linéaire U_f portant sur une fonction $f(x)$ continue entre a et b peut se mettre sous la forme

$$U_f = \lim_{n=\infty} \int_a^b K_n(y) f(y)\, dy,$$

où les $K_n(y)$ sont des fonctions continues entre a et b. Il nous sera utile d'observer qu'on peut même supposer que les fonctions K_n soient nulles en a et b. Il suffit de remplacer $K_n(y)$ par $K_n(y) + T_n(y)$ en appelant $T_n(y)$ une fonction continue nulle de $a + \epsilon_n$ à $b - \epsilon_n$ et égale à

$$\frac{y - a - \epsilon_n}{\epsilon_n} K_n(a) \qquad \text{et} \qquad \frac{b - \epsilon_n - y}{\epsilon_n} K_n(b)$$

dans les intervalles respectifs $(a,\ a + \epsilon_n)$, $(b - \epsilon_n,\ b)$. En prenant, par exemple

$$\epsilon_n = \frac{1}{n(|K_n(a)| + 1)(|K_n(b)| + 1)},$$

on voit que l'on a

$$\left| \int_{b-\epsilon_n}^{b} \frac{b\epsilon_n - y}{\epsilon_n} K_n(b)\, dy \right| = \frac{|K_n(b)|}{2n(|K_n(a)| + 1)(|K_n(b)| + 1)} < \frac{1}{2n}.$$

Par suite, l'intégrale

$$\int_{b-\epsilon_n}^{b} T_n(y)\, dy$$

tend vers 0 avec $1/n$ et il en est de même de

$$\int_a^b T_n(y)\, dy.$$

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† Voir Transactions of the American Mathematical Society, vol. 5 (1905), pp. 493–499. Nous conservons ici les mêmes notations.

On peut aussi supposer que les $K_n(y)$ soient des polynomes. En effet puisque ce sont des fonctions continues, on peut trouver pour chaque valeur de n un polynome $P_n(y)$ tel que l'on ait, entre a et b,

$$|P_n(y) - K_n(y)| < \frac{1}{n}.$$

Et l'on aura

$$\lim_{n=\infty} \int_a^b P_n(y) f(y)\,dy = \lim_{n=\infty} \int_a^b K_n(y) f(y)\,dy = U_f.$$

Mais on ne pourra pas toujours supposer en même temps que $P_n(y)$ soit nul en a et b.

La remarque précédente nous fournit un nouveau développement de l'opération linéaire la plus générale portant sur une fonction continue. En effet, posons

$$P_n(y) = a_0^n + a_1^n y + \cdots + a_{p_n}^n y^{p_n}.$$

On aura

$$U_f = \lim_{n=\infty} \left[a_0^n U_f^{(0)} + a_1^n U_f^{(1)} + \cdots + a_{p_n}^n U_f^{(p_n)} \right], \tag{21}$$

avec

$$U_f^{(r)} = \int_a^b y^r f(y)\,dy. \tag{22}$$

On peut obtenir un autre développement en s'appuyant sur ce que toute fonction continue entre 0 et 1 peut être représentée par le développement uniformément convergent

$$f(y) = \lim_{n=\infty} \left[R_{0,n}(y) f(0) + \cdots R_{r,n}(y) f\left(\frac{r}{n}\right) + \cdots + R_{n,n}(y) f\left(\frac{n}{n}\right) \right],$$

où les $R_{r,n}(y)$ sont certains polynomes déterminés indépendamment de la fonction f.* De sorte qu'en appliquant l'opération U terme à terme dans le cas où $a = 0$, $b = 1$, l'on aura

$$U_f = \lim_{n=\infty} \left[b_0^n f(0) + \cdots + b_r^n f\left(\frac{r}{n}\right) + \cdots + b_n^n f\left(\frac{n}{n}\right) \right], \tag{23}$$

les nombres

$$b_r^n = \int_a^b f(y) R_{r,n}(y)\,dy$$

restant indépendants de la fonction $f(y)$.

Cette représentation de l'opération U_f est intéressante en ce qu'elle donne un procédé pour calculer effectivement U_f connaissant seulement les valeurs de $f(y)$ en tous les points de l'ensemble E des points d'abscisses commensurables.

* Voir E. BOREL, *Leçons sur les fonctions de variables réelles*, p. 80, Gauthier-Villars, Paris, 1904.

On pourrait d'ailleurs l'étendre au cas où E est un ensemble dénombrable partout dense entre 0 et 1.

Mais les deux développements (21) et (23) ont l'inconvénient de ne pas être chacun seul de son espèce: autrement dit, les coefficients a_r^n, b_r^n ne sont pas déterminés d'une façon unique par l'opération U_f. Ainsi, par exemple, la formule (23) subsisterait en remplaçant les b_r^n par les quantités $b_r^n + d_r^n/n^2$ pourvu que les nombres arbitraires d_r^n soient bornés dans leur ensemble.

Au contraire, *le développement* (4′)* *en série indéterminée n'est valable que pour une suite de valeurs* $u_0, u_1, \cdots$ *bien déterminées par l'opération* U_f, comme on s'en assure facilement.

Décomposition d'une opération linéaire, correspondant à une décomposition de l'intervalle.

Soit c un nombre quelconque compris entre a et b et soient ϕ_1 et ϕ_2 deux fonctions continues définies de la façon suivante: $f(x)$ étant une fonction continue de a à b qui coïncide avec $f_1(x)$ de a à c et avec $f_2(x)$ de c à b, nous prendrons

$$\phi_1(x) = f(x) - \tfrac{1}{2} f(c) \text{ de } a \text{ à } c, \qquad \phi_1(x) = \tfrac{1}{2} f(c) \text{ de } c \text{ à } b,$$

puis

$$\phi_2(x) = f(x) - \phi_1(x).$$

Alors, on aura

$$U_f = U_{\phi_1} + U_{\phi_2}.$$

Or la fonction continue ϕ_1 est déterminée connaissant seulement la fonction $f_1(x)$ entre a et c. Donc U_{ϕ_1} peut être considérée comme une opération fonctionnelle V_{f_1} portant sur la fonction continue $f_1(x)$ définie entre a et c. De plus, on voit facilement que cette opération V_{f_1} est distributive et continue, c'est à dire linéaire. *On pourra ainsi écrire*

$$U_f \equiv V_{f_1} + W_{f_2}, \tag{24}$$

V et W étant deux opérations linéaires définies respectivement dans les intervalles (a, c), (c, b).

Étant donnée une telle décomposition, on peut en trouver immédiatement une infinité d'autres: il suffit de remplacer respectivement V_{f_1} et W_{f_2} par $V_{f_1} + Af_1(c)$ et $W_{f_2} - Af_2(c)$, A étant une constante arbitraire. Il n'y a pas d'autres décompositions. Autrement dit, *si* v_{f_1}, V_{f_1} *et* w_{f_2}, W_{f_2} *sont des opérations linéaires définies respectivement dans les intervalles* (a, c) *et* (c, b) *et telles que l'on ait*

$$U_f \equiv v_{f_1} + w_{f_2} \equiv V_{f_1} + W_{f_2},$$

on a nécessairement

$$v_{f_1} \equiv V_{f_1} + Af(c), \qquad w_{f_2} \equiv W_{f_2} - Af(c),$$

*Transactions, loc. cit., p. 496.

A étant une constante arbitraire. En effet, les deux membres de l'identité évidente

$$v_{f_1} - V_{f_1} = W_{f_2} - w_{f_2}$$

représentent une même opération qui ne varie pas quand on remplace f_1 et f_2 par deux fonctions continues quelconques ayant en c la valeur $f(c)$. Par conséquent, cette opération est une fonction de $f(c)$, au sens ordinaire: soit, $G[f(c)]$. Et comme on doit avoir

$$G[f(c)] + G[F(c)] = G[f(c) + F(c)],$$

G est nécessairement de la forme $Af(c)$.

Comme on a

$$U_f = \lim_{n=\infty} \int_a^b K_n(y) f(y)\, dy,$$

on serait tenté de dire que l'une des décompositions précédentes est fournie par les formules,

$$(25) \qquad v_{f_1} = \lim_{n=\infty} \int_a^c K_n f_1\, dy \qquad w_{f_2} = \lim_{n=\infty} \int_c^b K_n f_2\, dy.$$

Mais cette conclusion serait prématurée, car il n'est pas certain que les seconds membres de (25) soient convergents. Cependant, nous allons montrer qu'on peut toujours choisir les fonctions K_n *continues de* a *à* b de façon à ce qu'il en soit ainsi. En effet; soit

$$U_f = V_{f_1} + W_{f_2}$$

l'une des décompositions possibles de U_f. D'après notre première remarque, on peut trouver des fonctions $H_n(y)$, $L_n(y)$ respectivement continues dans (a, c) et (c, b) *mais nulles en* c, de façon que l'on ait

$$V_{f_1} = \lim_{n=\infty} \int_a^c H_n f_1\, dy, \qquad W_{f_2} = \lim_{n=\infty} \int_c^b L_n f_2\, dy.$$

Or la fonction $K_n(y)$ qui coïncide avec H_n de a à c et avec L_n de c à b *est continue de* a *à* b et telle que l'on ait

$$U_f = \lim_{n=\infty} \int_a^b K_n f dy,$$

—ce qui démontre la proposition.

On généralise immédiatement ce qui précède au cas où l'on diviserait l'intervalle (a, b) en un nombre fini quelconque d'intervalles contigus.

Etant donnée une opération fonctionnelle *continue* U_f quelconque portant sur la fonction $f(x)$ continue entre a et b, on ne peut pas toujours la mettre sous la forme d'une somme de deux opérations *continues* V_{f_1}, W_{f_2}, déterminées quand on connait les valeurs $f_1(x)$ et $f_2(x)$ de $f(x)$ dans les intervalles (a, c), (c, b).

Mais il ne faudrait pas croire cependant que les opérations *linéaires* sont les seules opérations continues qui jouissent de cette importante propriété. Soit par exemple, l'opération continue $U_f = [f(\alpha)]^2$ où α est un nombre fixe entre a et b. Appelons V_{f_1}, l'opération continue égale à $f_1^2(\alpha)$ si $\alpha \leqq c$, nulle si $\alpha > c$ et W_{f_2} l'opération continue égale à 0 dans le premier cas, à $f_2^2(\alpha)$ dans le second. On a $U_f = V_{f_1} + W_{f_2}$ et cependant l'opération U_f n'est pas linéaire.

Importance du champ fonctionnel dans lequel on définit une opération linéaire.

Nous avons supposé jusqu' ici que les opérations linéaires sur lesquelles nous raisonnions faisaient correspondre un nombre U_f à *toute* fonction $f(x)$ continue entre a et b et cela sans savoir si U_f était définie pour d'autres fonctions. Cette hypothèse est essentielle, comme nous allons le voir. Tout d'abord, observons que l'on peut construire des opérations linéaires définies pour toute fonction $f(x)$ *uniforme* entre a et b. Il suffit de prendre par exemple $U_f = f(a)$.

Pour donner un exemple plus général, considérons un ensemble dénombrable E de points entre a et b: $c_1, c_2, c_3, \cdots$ et soit

$$A_1 + A_2 + \cdots + A_n + \cdots$$

une série absolument convergente. L'opération

$$U_f = A_1 f(c_1) + \cdots + A_n f(c_n) + \cdots$$

sera définie et linéaire pour toute fonction $f(x)$ uniforme et bornée entre a et b.

Or, *on ne peut étendre à de telles opérations le théorème de* M. HADAMARD. En effet son énoncé même suppose qu'on peut calculer l'intégrale

$$\int_a^b K_n f dy;$$

il faut pour cela que f soit intégrable. Par conséquent, il ne peut avoir de sens que dans le champ des fonctions intégrables. Mais cette restriction n'est pas encore suffisante. *Étant donnée une opération linéaire U_f définie pour toute fonction sommable* $f(x)$ entre a et b, on ne peut pas toujours trouver une suite de fonctions mesurables et bornées $K_n(x)$ telles que l'on ait*

$$U_f = \lim_{n=\infty} \int_a^b K_n(y) f(y) dy. \tag{26}$$

(On voit que nous n'imposons même plus aux K_n la condition d'être continues). Il suffit pour le voir de prendre par exemple $U_f = f(a)$. Si l'on applique la formule (26) à la fonction sommable $f(x)$ qui est égale à 1 pour $x = a$ et nulle partout ailleurs, on arrive à une impossibilité.

* Voir LEBESGUE, *Leçons sur l'intégration*, p. 115.

D'ailleurs, il faut observer, inversement, que le théorème de M. HADAMARD n'est pas uniquement applicable dans le champ des fonctions continues; sa démonstration même est valable lorsque $f(x)$ n'a qu'un nombre fini de discontinuités de première espèce, et c'est dans ce cas général qu'il avait été énoncé par son auteur.

Un autre genre de difficultés se présente quand on considère des opérations linéaires qui ne sont pas définies dans tout le champ des fonctions continues, comme l'opération $U_f = f'_c$. On peut être ainsi amené à considérer ce que nous appellerons des *opérations linéaires d'ordre n*, c'est à dire des opérations distributives portant seulement sur les fonctions ayant leurs n premières dérivées continues et telles que U_f ait pour limite U_ϕ lorsque *f et ses n premières dérivées* tendent uniformément vers ϕ et ses n premières dérivées.

Si U_f est une telle opération, le théorème de M. Hadamard lui est applicable et on peut même mettre U_f sous la forme suivante

$$U_f = A_0 f(a) + A_1 f'_a + \cdots + A_{n-1} f^{(n-1)}_a + \lim_{p=\infty} \int_a^b K_p f^{(n)} dy,$$

les A_i désignant certaines constantes et les K_p certaines fonctions continues de a à b. Il suffit pour le voir, d'appliquer l'opération U_f à chaque terme de l'identité

$$f(x) \equiv f(a) + (x-a) f'_a + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}_a + \int_a^x \cdots \int_a^x f^{(n)}(x) (dx)^n.$$

Au contraire, si l'on décompose (a, b) en deux intervalles, le raisonnement qui nous a conduit à établir la formule (24) n'est plus valable, car les fonctions ϕ_1 et ϕ_2 ne sont pas dérivables jusqu'à l'ordre n. Mais le résultat est encore exact, si nous prenons de a à c

$$\phi_1(x) = f(x) - \tfrac{1}{2}\left[f(c) + (x-c) f'_c + \cdots + \frac{(x-c)^n}{n!} f^{(n)}_c \right]$$

et de c à b

$$\phi_1(x) = \tfrac{1}{2}\left[f(c) + (x-c) f'_c + \cdots + \frac{(x-c)^n}{n!} f^{(n)}_c \right];$$

puis $\phi_2 \equiv f - \phi_1$.

Seulement, si l'on a, entre des opérations linéaires d'ordre n, la relation

$$v_{f_1} + w_{f_2} \equiv V_{f_1} + W_{f_2},$$

on n'aura plus nécessairement

$$v_{f_1} - V_{f_1} = A f(c),$$

mais

$$v_{f_1} - V_{f_1} = A_0 f(c) + A_1 f'_c + \cdots + A_n f^{(n)}_c,$$

$A_0, A_1, \cdots, A_n$ désignant certaines constantes.

Equations fonctionnelles. Il peut arriver que l'on ait à chercher les fonctions $\theta(x)$ telles que U_θ prenne une valeur donnée A. La solution est bien simple. Si nous supposons en effet, l'opération U non identiquement nulle dans le champ H des fonctions auxquelles elle s'applique, il existe au moins une fonction $\psi(x)$ de ce champ, non identiquement nulle entre a et b et telle que U_ψ soit différent de zéro. Il y a au moins une abscisse c de l'intervalle (a, b) qui n'annule pas $\psi(x)$. Ceci étant, il est extrèmement facile de voir qu'*on obtiendra toutes les racines* $\theta(x)$ *de l'équation* $U_\theta = A$, *chacune une fois et une seule, par la formule*

$$\theta(x) = f(x) + \frac{A - U_f}{U_\psi}\psi(x),$$

où $f(x)$ *est remplacée successivement par toutes les fonctions distinctes, nulles en* c, *du champ* H. Le raisonnement s'applique aussi bien quand le champ H est celui des fonctions continues entre a et b qu'aux opérations linéaires d'ordre quelconque.

PARIS, *Novembre*, 1904.

SURFACES WHOSE GEODESICS MAY BE REPRESENTED IN THE PLANE BY PARABOLAS*

BY

EDWARD KASNER

The general problem of geodesic representation, whose preliminary discussion was given by the author in the Transactions for 1903,† is essentially the the same as what may be termed the *inverse problem of geodesics*, proposed by LIE‡ as follows: Given the equation of the geodesics in the form $F(u, v, \lambda, \mu) = 0$, determine the corresponding surfaces.

For the existence of such a surface, it is necessary that the differential equation of given system should be of the form

$$y'' = Ay'^3 + By'^2 + Cy' + D, \tag{1}$$

where the coefficients are functions of x, y. We shall refer to any equation of this form as one of *cubic type*. Such equations are of interest independently of their connection with geodesics: the type is invariant with respect to arbitrary point transformation.

In order that (1) shall represent a system of possible geodesics, it is further necessary that the functions A, B, C, D satisfy certain relations expressed by the consistency of a system of partial differential equations.§ For our purpose, it is not necessary to have these relations explicitly. We note merely that the equations of *geodesic type* constitute only a subclass of the general cubic class (1)—a subclass which is itself invariant under the group of all point transformations.

A more special subclass is formed by those equations of the second order which are equivalent, under point transformation, to $y'' = 0$. Such an equation may be characterized by the fact that its complete integral can be put into the form

$$f_1(x, y) + \text{const.}\, f_2(x, y) + \text{const.}\, f_3(x, y) = 0,$$

* Presented to the Society December 30, 1902. Received for publication February 25, 1905.

† *The generalized Beltrami problem concerning geodesic representation*, vol. 4, pp. 149–152.

‡ *Untersuchungen über geodetischen Curven*, Mathematische Annalen, vol. 20 (1882), in particular p. 445.

§ Cf. R. LIOUVILLE, *Sur la caractère auquel se reconnaît l'équation différentielle d'un système géodésique*, Comptes Rendus, vol. 108 (1889), p. 495.

of the first degree in the two parameters. This subclass includes then all linear systems, in the general sense of the term. It is a part of the geodesic class, corresponding, by BELTRAMI's theorem, to the surfaces of constant curvature.* The necessary and sufficient conditions on the coefficients of (1), for such a system, are †

$$(BD - D_y)_y - (AD + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y)_x + C(AD + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y) - D(AC + A_x) = 0,$$

$$-(AC + A_x)_x + (AD + \tfrac{2}{3}B_x - \tfrac{1}{3}C_y)_y + B(AD + \tfrac{2}{3}B_x - \tfrac{1}{3}C_y) - A(BD - D_y) = 0.$$

In the present paper we consider problems of this kind: Given a system of plane curves involving three or more parameters, determine the two-parameter systems capable of representing the geodesics of a surface, and investigate the corresponding surfaces. A more general problem is the determination of all the integrals of form (1) satisfying a given differential equation of the third or higher order.‡ The only case that has received treatment is the case of circles; here the only systems of geodesic type are linear, so that the corresponding surfaces are merely those of constant curvature. §

The main part of the paper, §§ 1–7, is devoted to what seems to be the simplest case of our problem, namely, the three-parameter system of vertical parabolas

$$y = \lambda x^2 + \mu x + \nu, \tag{2}$$

or the corresponding differential equation

$$y''' = 0. \tag{3}$$

We shall find that this leads to interesting and important results; in particular, it is the first case which leads to surfaces other than those of constant curvature.

In § 1, we find all the doubly infinite systems of parabolas whose differential equation is of the cubic form (1). In addition to linear systems, certain quadratic systems satisfy the conditions of the problem (§ 2). The systems are further classified and characterized geometrically (§ 3, § 4). In § 5 the existence

* Cf. Transactions, vol. 4 (1903), p. 119.

† R. LIOUVILLE, Journal de l'Ecole Polytechnique, vol. 57 (1887), p. 219. For an application of these equations cf. E. KASNER, *A characteristic property of isothermal systems*, Mathematische Annalen, vol. 59 (1904), p. 352.

‡ In general no solution exists, since the functions A, B, C, D must satisfy an infinite number of conditions. An example where the inconsistency of the conditions may be verified very simply is $y''' = y^6$. The same holds for equations of higher order.

§ Cf. BUSSE, Berliner Sitzungsberichte, 1896. It may be shown, by an investigation similar to that in § 1 of the present paper, that the only systems of circles of the cubic type (1) are the linear systems.

of corresponding surfaces is verified, namely, those of constant curvature, and those which are geodesically representable on surfaces with the linear element

$$ds^2 = v(du^2 + dv^2).$$

In § 7 the group of the geodesics is determined in order to obtain the most general representation in which geodesics are pictured by parabolas. In § 6 the problem is specialized by adding the requirement that the representation shall be conformal; the surfaces are then either developables or applicable on a certain surface of revolution. Finally, in § 8, certain more general problems in conformal representation are treated; in this section the discussion is merely outlined since no new classes of surfaces are brought to light.

The surfaces of variable curvature which satisfy the conditions of our problem present themselves, in whole or in part, in many important investigations. We mention here the following, detailed references being given later: (1) Systems of geodesics admitting infinitesimal transformations (LIE, KOENIGS). (2) Systems of geodesic circles admitting contact transformations (LIE). (3) Surfaces for which the partial differential equation $\Delta\theta = 1$ admits integrals of the first and second degree in the first derivatives (DARBOUX). (4) Surfaces whose element is reducible to the LIOUVILLE form in an infinite number of ways (RAFFY, KOENIGS).

§ 1. *Differential equations of the problem.*

We first obtain the conditions that an equation of cubic type (1) shall represent a double infinity of parabolas (2). For this purpose we differentiate (1) and find that

$$y''' = \alpha_1 y'^5 + \alpha_2 y'^4 + \alpha_3 y'^3 + \alpha_4 y'^2 + \alpha_5 y' + \alpha_6, \tag{4}$$

where

$$\begin{aligned} \alpha_1 &= 3A^2, & \alpha_4 &= 3AD + 3BC + B_x + C_y, \\ \alpha_2 &= 5AB + A_y, & \alpha_5 &= 2BD + C^2 + C_x + D_y, \\ \alpha_3 &= 4AC + 2B^2 + A_x + B_y, & \alpha_6 &= CD + D_x. \end{aligned} \tag{5}$$

Since for our parabolas equation (3) must be satisfied, we have then the conditions

$$\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0,\ \alpha_4 = 0,\ \alpha_5 = 0,\ \alpha_6 = 0.$$

The first of these shows that

$$A = 0; \tag{6}$$

the second is then satisfied identically; and the others give the following four equations for the determination of B, C, D:

$$
\begin{aligned}
B_y + 2B^2 &= 0, \\
B_x + C_y + 3BC &= 0, \\
C_x + D_y + C^2 + 2BD &= 0, \\
D_x + CD &= 0.
\end{aligned} \tag{7}
$$

It will be convenient to divide the discussion of this system with reference to the vanishing or non-vanishing of B.

Discussion for $B = 0$.

Putting $B = 0$ into (7_2), we find that C is a function of x alone, say the function X. The remaining equations of the system then give

$$D_y + X' + X^2 = 0, \qquad D_x + XD = 0. \tag{8}$$

Eliminating D, we obtain

$$X'' + 3XX' + X^3 = 0,$$

whose solution is

$$C = X = \frac{2(ax + b)}{ax^2 + 2bx + c}.$$

The value of D is now found, from (7_4), to be

$$D = \frac{-2ay + 2d}{ax^2 + 2bx + c}.$$

These values of C and D, together with $A = 0$, $B = 0$, satisfy all our conditions. The corresponding equation (1) is

$$(ax^2 + 2bx + c)y'' = 2(ax + b)y' - 2(ay - d). \tag{9}$$

Discussion for $B \neq 0$.

The integration of (7_1) gives

$$B = (2y + X)^{-1}, \tag{10}$$

where X is an arbitrary function of x. We now simplify the remaining equations of the set (7) by introducing new variables u, v as follows:

$$u = x, \qquad v = (2y + X)^{-1}. \tag{11}$$

The result of the transformation, noting that $B = v^2$, is found to be

$$
\begin{aligned}
vC_v - 3C + U'v^2 &= 0, \\
-v^3 D_v + 2v^2 D + C^2 + C_u - \tfrac{1}{2}U'v^3 C_v &= 0, \\
D_u - \tfrac{1}{2}U'v^3 D_v + CD &= 0,
\end{aligned} \tag{12}
$$

where U denotes the result of substituting u for x in the function X.

The solution of the first of these equations is

$$C + U'v^2 + U_1v^3, \tag{13}$$

where U_1 is an arbitrary function of u. The other equations may now be written

$$\begin{aligned} D_u &= -U_1v^3D + \tfrac{1}{2}U'v^2R, \\ D_v &= 2v^{-1}D + v^{-1}R, \end{aligned} \tag{14}$$

where R is defined by

$$R = U'' + U_1'v + \tfrac{1}{2}U'U_1v^3 + U_1^2v^4. \tag{14'}$$

Equating the two values of D_{uv} found from (14), we have

$$3U_1D + S = 0, \tag{15}$$

where

$$S = U_1R - \tfrac{1}{2}U'R_v + v^{-3}R_u. \tag{15'}$$

We now proceed to show that if U_1 does not vanish, the equations (14) are inconsistent. For, under the assumption that $U_1 \neq 0$, (15) may be written

$$D = -\frac{S}{3U_1}.$$

Substituting this in (14_2), we find

$$S_v - 2v^{-1}S + 3U_1v^{-1}R = 0.$$

This must be satisfied identically if equations (14) are to be consistent. Substituting the values of R and S given in (14') and (15'), the coefficient of every power of v must vanish. For our purpose it is sufficient to select the coefficient of v^3, which is found to be $5U_1^3$. This, however, cannot vanish, in virtue of the assumption made at the outset.

We may therefore assume that

$$U_1 = 0, \tag{16}$$

Then equation (15) gives $S = 0$. From (14'), R reduces to U''; and thereby, from (15'), S reduces to $v^{-3}U'''$. Equating the two values of S thus obtained, we have $U''' = 0$, or

$$U = au^2 + 2bu + c, \tag{17}$$

where a, b, c are arbitrary constants.

Substituting the values of U and U_1 in (14), we find

$$D_u = 2av^2(au + b), \qquad D_v = 2b^{-1}(D + a).$$

From the derivation these equations must be consistent; their common solution, containing a new arbitrary constant d', is in fact

$$D = av^2(au^2 + 2bu + d') - a. \tag{18}$$

The value of C given in (13) reduces, in virtue of (16) and (17), to

$$C = 2v^2(au + b). \tag{19}$$

This completes the discussion of the set (7) for $B \neq 0$. The solution found, given in (10), (19), (18), may be reduced by means of the substitution formulas (11) and the value

$$X = ax^2 + 2bx + c,$$

to the form

$$B = \frac{1}{2y + ax^2 + 2bx + c},$$

$$C = \frac{2(ax + b)}{2y + ax^2 + 2bx + c}, \tag{19}$$

$$D = \frac{-2(ay - d)}{2y + ax^2 + 2bx + c},$$

where d takes the place of $a(d' - c)$.

The differential equation (1), thus found, is

$$(2y + ax^2 + 2bx + c)y'' = y'^2 + 2(ax + b)y' - 2(ay - d). \tag{20}$$

We may state the result obtained at this stage as follows:

The only equations of cubic type (1) *which represent a doubly infinite system of parabolas* (2) *are those of the forms* (9) *and* (20). *Both these forms are included in the complete solution*

$$(2ty + ax^2 + 2bx + c)y'' = ty'^2 + 2(ax + b)y' - 2(ay - d) \tag{21}$$

containing four constants $a : b : c : d : t$.

§ 2. *General character of the systems of parabolas.*

There is no difficulty in integrating (21); for it is known, from the derivation, that the integral is of the form (2). We find that *our systems of parabolas are represented by*

$$y = \lambda x^2 + \mu x + \nu,$$

where λ, μ, ν *are connected by a relation of the form*

$$t(\mu^2 - 4\lambda\nu) - 2c\lambda + 2b\mu - 2a\nu + 2d = 0. \tag{21'}$$

The character of the system, we shall see, depends essentially upon the discriminant of this quadratic in λ, μ, ν, namely,

$$\delta = 4t^2(b^2 - ac - 2td). \tag{22}$$

We show first that, if $\delta = 0$, the equation (21) is equivalent, under the group of point transformations, to $y'' = 0$. This requires the discussion of the cases $t = 0$ and $b^2 - ac - 2td = 0$.

If $t = 0$, that is, if equation (21) is of the form (9), the corresponding relation (21′) reduces to the form

$$-2c\lambda + 2b\mu - 2a\nu + 2d = 0. \tag{9′}$$

This system of parabolas being linear, is evidently reducible to $y'' = 0$.

If t is not 0, we may assume it to be unity, so that our equation is of form (20). The vanishing of δ requires $b^2 - ac - 2d = 0$. The transformation

$$x_1 = x, \qquad 2y_1 = 2y + ax^2 + 2bx + c \tag{23}$$

then converts (20) into

$$2y_1 y_1'' = y_1'^2, \tag{24}$$

For this normal form the relation (21′) becomes

$$\mu^2 - 4\lambda\nu = 0.$$

We may therefore put

$$\lambda = \alpha^2, \qquad \mu = 2\alpha\beta, \qquad \nu = \beta^2,$$

so that the solution of (24) may be written

$$y = (\alpha x + \beta)^2. \tag{24′}$$

This represents the vertical parabolas touching the axis of x. In the theory of algebraic curves this would be termed a quadratic system since the parameters are involved to the second degree. But in the general (infinitesimal) theory * of systems of curves the system (24′) is linear since it may be written

$$\alpha x + \beta - \sqrt{y} = 0,$$

which involves the parameters linearly. Hence the equation (24) is equivalent to $y'' = 0$. This completes the prooof of the result stated above.

We pass now to the case $\delta \neq 0$. We take, as before, $t = 1$, and put

$$\kappa^2 = ac - b^2 + d; \tag{25}$$

so that κ, by assumption, does not vanish. The transformation †

$$x_1 = x, \qquad 2\kappa y_1 = 2y + ax^2 + 2bx + c \tag{26}$$

converts (20) into

$$2y_1 y_1'' = y_1'^2 + 1. \tag{27}$$

*See Bulletin of the American Mathematical Society, vol. 12 (1905), p. 307.

† If reality considerations are taken into account, it is necessary to separate the cases $\delta < 0$, and $\delta > 0$. In the latter case, κ is imaginary; but another transformation gives the normal form $2yy'' = y'^2 - 1$. This represents the vertical parabolas whose foci are on the axis of x.

The solution for this normal form is

$$4\beta(y-\beta) = (x-\alpha)^2,$$

which represents the parabolas having the axis of x for directrix.

This is an essentially distinct case; the equation (27) is not equivalent to $y''=0$, since it does not satisfy the conditions of LIOUVILLE given in the introduction. We have then this result:

With respect to the group of all point transformation, the systems of parabolas possessing a cubic differential equation (1), *that is, the systems* (21), *divide into two distinct types. In the first type, the discriminant* δ *of the relation* (21′) *vanishes, and the system is linear, i. e., equivalent to* $y''=0$. *In the second type,* δ *does not vanish, the system is essentially quadratic, and equivalent to* (27).

§ 3. *Detailed classifications.*

The preceding paragraph contains the classification of our systems (21) with respect to the group of all point transformations. We can obtain a closer geometric survey of the systems by means of classifications based on certain finite (continuous) groups which suggest themselves in connection with our problem.

The totality of vertical parabolas (2) evidently admits the projective transformations of the form

$$G_5: \qquad x_1 = e_1x + e_2, \qquad y_1 = e_3y + e_4x + e_5,$$

which constitute a five-parameter group. It may be shown that it admits a more extensive group

$$G_6: \qquad x_1 = e_1x + e_2, \qquad y_1 = e_0x^2 + e_3y + e_4x + e_5.$$

This group G_6 contains G_5 as a subgroup, the latter arising by putting $e_0 = 0$.

Since any point transformation converts an equation of cubic form (1) into an equation of the same form, and since the particular transformations G_6 also convert vertical parabolas into vertical parabolas, it follows that the transformations G_6 convert any solution of our problem, that is, any system (21), into a solution or system (21). The effect of a transformation G_6 upon the equation (21) containing the constants $t:a:b:c:d$, is to convert it into an equation of the same form with new constants $t_1:a_1:b_1:c_1:d_1$.

The induced transformation of the constants is found to be

$$G_6': \begin{cases} t_1 = e_3^2 t, \\ a_1 = 2e_3e_0t + e_3e_1a, \\ b_1 = e_3e_4t + e_3e_1e_2a + e_3e_1b, \\ c_1 = 2e_3e_5t + e_3e_2^2a + 2e_3e_2b + e_3c, \\ d_1 = \dfrac{e_4^2 - 4e_0e_5}{2}t + (e_1e_2e_4 - e_0e_2^2 - e_1^2e_5)a + (e_1e_4 - 2e_0e_2)b - e_0c + e_1^2d. \end{cases}$$

The induced group G_5', corresponding to the projective transformations G_5, is obtained by simply putting $e_0 = 0$ in the above. The modulus of the linear transformation is $e_1^5 e_3^3$; this cannot vanish, since otherwise the original transformation G_6 would have a vanishing jacobian and thus degenerate. We may therefore assume that neither e_1 nor e_3 vanishes.

We are now in position to discuss the equivalence of our systems of parabolas, or the corresponding differential equations (21), with respect to the group G_6. From the importance of the quantity δ in the general discussion of § 2, one is led to expect its invariant character in the present connection. In fact, the factors of δ give rise to two (relative) invariants t and $b^2 - ac - 2td$; for

$$t_1 = e_3^2 t,$$
$$b_1^2 - a_1 c_1 - 2t_1 d_1 = e_1^2 e_3^2 (b^2 - ac - 2td).$$

This shows that the systems for which $t = 0$ must be distinguished from those for which $t \neq 0$; and, similarly, the cases $b^2 - ac - 2td = 0$ and $b^2 - ac - 2td \neq 0$ are essentially distinct. It is also observed, from the form of G_6' above, that when t vanishes, $a_1 = e_3 e_1^2 a$, so that it is necessary then to distinguish further according to the vanishing or non-vanishing of a. *We are thus led to the following classification of the systems* (21) *with respect to* G_6:

$$\begin{aligned}
&\text{I:} && t = 0,\ b^2 - ac - 2td = 0,\ a = 0;\\
&\text{II:} && t = 0,\ b^2 - ac - 2td = 0,\ a \neq 0;\\
&\text{III:} && t = 0,\ b^2 - ac - 2td \neq 0,\ a = 0;\\
&\text{IV:} && t = 0,\ b^2 - ac - 2td \neq 0,\ a \neq 0;\\
&\text{V:} && t \neq 0,\ b^2 - ac - 2td = 0;\\
&\text{VI:} && t \neq 0,\ b^2 - ac - 2td \neq 0.
\end{aligned}$$

No member of one of these classes can be transformed into a member of another class; while systems belonging to the same class are equivalent.

The last statement may be proved by showing that for each class it is possible to set up a canonical form to which all the members of that class are equivalent. We omit detailed discussion and merely state the *canonical equations with their solutions*: *

$$\begin{aligned}
&\text{I:} && y'' = 2, && y = x^2 + \alpha x + \beta;\\
&\text{II:} && x^2 y'' = 2xy' - 2y, && y = \alpha x^2 + \beta x;\\
&\text{III:} && xy'' = y', && y = \alpha x^2 + \beta;\\
&\text{IV:} && (x^2 - 1)y'' = 2xy' - 2y, && y = \alpha x^2 + \beta x + \alpha;\\
&\text{V:} && 2yy'' = y'^2, && y = (\alpha x + \beta)^2;\\
&\text{VI:} && 2yy'' = y'^2 + 1, && 4\beta y = (x - \alpha)^2 + 4\beta^2.
\end{aligned}$$

* The discussion for the cases where $t \neq 0$ is really contained in § 2. The transformations there employed, namely (23) and (26), belong to G_6. Hence (24) and (27) may be used as the canonical equations for V and VI respectively.

By a transformation of the type G_6 any system (21) can be reduced to one of these forms.

In classes I–IV, the relation (21′) between λ, μ, ν is of the first degree, while in classes V, VI the relation is of the second degree. Furthermore, classes I–V all belong to the first type described in the final theorem of § 2, while VI is identical with the second type.

We pass now to the projective group G_5. In the formulas for G_6', we put then $e_0 = 0$, so that a is a (relative) invariant as well as t and $b^2 - ac - 2td$. The six classes considered above are of course distinct with respect to the smaller group now being considered; but V and VI each subdivide according to the vanishing or non-vanishing of a. A detailed discussion shows that the only other subdivision takes place in I; here $d = 0$ must be separated from $d \neq 0$.

We then have the following complete classification of the equations (21) *with reference to the projective group* G_5:

I_0: this is I with $d = 0$, canonical form $y'' = 0$;

I_1: this is I with $d \neq 0$;

II, III, IV: same as in the G_6 classification;

V_1: this is V with $a = 0$;

V_2: this is V with $a \neq 0$, canonical form $(2y + x^2)y'' = y'^2 + 2xy' - 2y$;

VI_1: this is VI with $a = 0$;

VI_2: this is VI with $a \neq 0$, canonical form $(2y + x^2)y'' = y'^2 + 2xy' - 2y + 1$.

As canonical forms for I_1, II, III, IV, V_1, VI_1, we may take the forms for I, II, III, IV, V, VI given in the G_6 classification above. The case I_0 might be omitted as trivial since the system does not consist of proper parabolas but of straight lines. These enter into the discussion since straight lines, as well as vertical parabolas, satisfy the equation $y''' = 0$.

In the first five of the nine classes the relation between λ, μ, ν is linear. Hence: *Of the two parameter systems of curves* $y = \lambda x^2 + \mu x + \nu$ *defined by linear relations*

$$c\lambda - b\mu + a\nu - d = 0,$$

there are five projectively distinct cases:

$$\begin{aligned}
&I_0: && a = 0,\ b = 0,\ d = 0;\\
&I_1: && a = 0,\ b = 0,\ d \neq 0;\\
&II: && a \neq 0,\ b^2 - ac = 0;\\
&III: && a = 0,\ b^2 - ac \neq 0;\\
&IV: && a \neq 0,\ b^2 - ac \neq 0.
\end{aligned}$$

The first consists merely of straight lines, the others of proper parabolas. Systems belonging to the same class are equivalent under the projective transformations G_5.

§ 4. *Geometric determinations.*

In this section we characterize geometrically the systems (21) and the various classes discussed in the preceding section. The discussion is merely outlined, but the results are complete.

Consider first the systems where $t = 0$, i. e., the systems where the relation (21′) takes the linear form (9). Such a linear relation is obtained by taking a fixed parabola* $y = \lambda_0 x^2 + \mu_0 x + \nu_0$ and a variable parabola $y = \lambda x^2 + \mu x + \nu$; the straight line through their two points of intersection is

$$(\lambda - \lambda_0)y + (\lambda_0\mu - \mu_0\lambda)x + \lambda_0\nu - \nu_0\lambda = 0;$$

and the condition that this line passes through a fixed point (x_0, y_0) is

$$(y_0 - \mu_0 x_0 - \nu_0)\lambda + \lambda_0 x_0 \mu + \lambda_0 \nu - \lambda_0 y_0 = 0;$$

which is of the required form. Hence

Any system defined by a linear relation between λ, μ, ν *may be characterized as consisting of the* ∞^2 *parabolas intersecting a fixed parabola* π_0 *in pairs of points collinear with a fixed point* P_0.

The classes described at the close of § 3 (omitting I_0) are distinguished as follows: In cases II and IV the point P_0 is finite, while in I and III it is at infinity. In I and II the point P_0 is on the parabola π_0, while in III and IV it is not so situated.

We pass now to the systems for which $t \neq 0$, i. e., the systems for which the relation (21′) is really quadratic. To characterize these we make use of the canonical forms V and VI in § 3, and the transformations G_5 and G_6.

For V, the canonical equation represents the parabolas tangent to the axis of x. The transformation G_5 converts the axis into some straight line; and a quadratic transformation G_6 converts the axis into some parabola. Therefore

Any system defined by a quadratic relation (21′) *with vanishing discriminant consists of the* ∞^2 *parabolas tangent to either a fixed straight line* (*case* V_1) *or a fixed parabola* (*case* V_2).

For VI, the canonical equation represents the parabolas having the axis of x for directrix. If a point is taken on this axis, the pairs of tangents drawn to the members of the system constitute an involution. Applying a transformation G_5, the axis of x is converted into some line and the involution property still holds for any point of this line. The effect of a transformation G_6, on the other hand, is to replace the axis of x by a parabola, and the pencil of lines through the point on the axis by a pencil of congruent parabolas through the corresponding point. The result is:

Any system defined by a quadratic relation (21′) *with non-vanishing discriminant may be obtained as follows: Take a pencil of straight lines* (*case* VI_1), *or a*

* The term parabola is used throughout in the sense of vertical parabola.

pencil of congruent parabolas (case VI_2*), and establish an involution therein. The* ∞^2 *parabolas touching the pairs (of straight lines or parabolas) belonging to the involution constitute the required system.*

§ 5. *The corresponding surfaces.*

As was remarked in the introduction, the fact that an equation is of the cubic form (1) does not of itself show that it represents the geodesics of a surface. But for the systems of parabolas obtained we shall find that corresponding surfaces do exist.

At the close of § 2, it was seen that with respect to point transformation the systems are of two distinct types. In the first type, the equation is equivalent to $y'' = 0$; so that, by the familiar BELTRAMI theorem, the surfaces are merely those of constant curvature.

It remains then to discuss only the second type, whose canonical form is

$$2yy'' = y'^2 + 1. \tag{27}$$

To find a corresponding surface, it is sufficient to note that, when the element of length of a surface is written in the isothermal form

$$ds^2 = E(du^2 + dv^2), \tag{28}$$

the differential equation of the geodesics is

$$v'' = -L_u v'^3 + L_v v'^2 - L_u v' + L_v, \tag{29}$$

where

$$L = \tfrac{1}{2} \log E.$$

This becomes of the same form as (27) if L is $\frac{1}{2} \log v$. The corresponding element is then

$$ds^2 = v(du^2 + dv^2). \tag{30}$$

Hence if the element of length of a surface can be reduced to the form (30), and the point (u, v) of the surface be represented by the point $x = u$, $y = v$ in the plane, then the geodesics of the surface are pictured by the parabolas (27) having the axis of x for directrix.

If two surfaces S and S_1 can be represented upon a plane so that the geodesics are pictured by the same curves, it follows that it is possible to establish a point-to-point correspondence between S and S_1 so that the geodesics correspond. When this is true we may say that S and S_1 are geodesically equivalent. The problem of finding all pairs of geodesically equivalent surfaces was solved by DINI and LIE. The only cases where S_1 is not applicable on (isometric to) S, or a surface homothetic to S, arise for the LIOUVILLE surfaces, whose elements are reducible to the form

$$ds^2 = \{U(u) + V(v)\}(du^2 + dv^2).$$

The element (30), obtained above, is of this type, hence there exist essentially distinct classes of surfaces with the same geodesic representation. It happens that LIE, in illustration of the general theory due to DINI and himself, has given the discussion of a particular case equivalent to (30). Through an oversight, however, LIE overlooked one form of solution.*

The resulting surfaces coincide, as KOENIGS has shown, with those surfaces whose geodesic equation, $\Delta\theta = 1$, admits precisely three homogeneous integrals of the second degree. Another equivalent problem is the determination of surfaces applicable on surfaces of revolution and having an element reducible to the LIOUVILLE form in an infinite number of ways. We may therefore make use of the forms obtained by RAFFY in his discussion of this question.†

If a surface can be represented geodesically upon a surface with the element (30), then its element is reducible to (30), or to one of the forms

$$ds^2 = \left(\frac{P}{v^2} + Q\right)(du^2 + dv^2), \tag{31}$$

$$ds^2 = (Pe^v + Qe^{2v})(du^2 + dv^2), \tag{32}$$

$$ds^2 = \frac{P + Q(e^v + e^{-v})}{(e^v - e^{-v})^2}(du^2 + dv^2); \tag{33}$$

so that (30), (31), (32), (33) together define a complete class of geodesically equivalent surfaces.

Applying this to our problem, we have the final result:

The only surfaces which can be represented point by point upon a plane so that the geodesics are pictured by parabolas $y = \lambda x^2 + \mu x + \nu$ are:

1°. *Surfaces of constant curvature. The system of parabolas is then linear (in the general sense), defined by a relation* (21′) *with vanishing discriminant* δ.

2°. *Surfaces of variable curvature whose linear element is reducible to one of the forms* (30), (31), (32), (33). *These are all geodesically equivalent; and are applicable on surfaces of revolution. The corresponding system of parabolas is essentially quadratic, defined by a relation* (21′) *whose discriminant does not vanish.*

There is no difficulty in finding the explicit equations of the surfaces of revolution whose elements are of the above form. We give the result for the simplest case. The class represented by (30) is applicable on any one of the surfaces of revolution

$$x = h\sqrt{v}\cos\frac{u}{h}, \qquad y = h\sqrt{v}\sin\frac{u}{h}, \qquad z = \int\sqrt{\frac{4v^2 - h^2}{4v}}\,dv, \tag{34}$$

* KOENIGS, *Mémoire sur les lignes géodésiques*, Mémoires des Savants Etrangeres, vol. 31 (1894), p. 22.

† RAFFY, Comptes Rendus, vol. 108 (1889), p. 493; DARBOUX, *Théories des surfaces*, vol. 3 (1894), p. 39.

where h is an arbitrary constant. The equation of the meridian curve is

$$z = h^3 \int \sqrt{4x^4 - h^6}\, dx. \tag{34'}$$

Another geometric characterization of our surfaces may be obtained by making use of WEINGARTEN's theorem: any surface applicable on a surface of revolution (exclusive of the catenoid) is a nappe of the evolute surface of a W-surface. The following simple result is obtained without difficulty. Take a surface whose principal radii are connected by the relation

$$R_1 + 2R_2 = 0,$$

and construct the nappe of its evolute corresponding to the radius R_1.* This gives surface with the element (30).†

§ 6. *Conformal representation.*

Having investigated all cases where a surface is capable of representation upon a plane so that the geodesics correspond to parabolas, we now determine those cases where the representation is conformal.

For this purpose, we recall that the most general conformal representation of any surface on a plane is obtained by putting ds^2 into the isothermal form (28) and setting $x = u$, $y = v$. The plane curves corresponding to the geodesics are then defined by an equation of the form

$$y'' = -L_x y'^3 + L_y y'^2 - L_x y' + L_y. \tag{35}$$

Comparing this with (1), we may state the useful

LEMMA. *In order that a system of plane curves shall correspond by conformal representation to the geodesics of a surface, it is necessary and sufficient that the equation of the system be of the cubic form*

$$y'' = Ay'^3 + By'^2 + Cy' + D, \tag{1}$$

and that the coefficients satisfy the relations

$$A = C, \qquad B = D, \qquad A_y + B_x = 0. \tag{36}$$

The only equations (21) which satisfy these conditions are found to be

$$(2ty + c)y'' = t(y'^2 + 1). \tag{37}$$

If $t = 0$, this equation reduces to $y'' = 0$, so that the system consists of straight lines. The only surfaces whose geodesics can be represented conformally by straight lines are, it is known, the developable surfaces.

* The other nappe has an element of the form $ds^2 = v^4(du^2 + dv^2)$.

† A more general theorem is given by LIE, Mathematische Annalen, vol. 20 (1882), p. 389.

If $t \neq 0$, the equation, by a translation of the x axis, may be reduced to (27). The corresponding surfaces are then represented by (30). Our result is therefore:

If a surface is capable of being represented conformally upon a plane in such a manner that its geodesics are pictured by parabolas $y = \lambda x^2 + \mu x + \nu$, *then either (case* 1°*) the surface is developable, and the representing curves are straight lines; or (case* 2°*) the surface belongs to the class* (30) *of surfaces applicable on the surface of revolution* (34), *and the representing curves are parabolas with a common directrix.*

§ 7. *Groups and representations.*

We have obtained all the systems of parabolas and all the surfaces connected with our problem. It remains now to determine all the possible representations.

Consider first the case of surfaces of constant curvature. The geodesics of such a surface, being equivalent to straight lines $y'' = 0$, admit an eight-parameter group, isomorphic to the projective group of the plane. Hence, for a given surface and a given system of parabolas belonging to the first type described at the close of § 2, there are ∞^8 possible representations. The total number of systems of this type is, however, ∞^3. Therefore:

A surface of constant curvature may be represented upon the plane so that its geodesics are pictured by parabolas (2) *in* ∞^{11} *ways. None of these representations is conformal except when the curvature is zero; then there are* ∞^4 *conformal representations.*

The corresponding discussion for the second type of surfaces obtained (those of variable curvature) is not so simple. We have seen that the system of parabolas may then be put into the canonical form (27). Our first step is to examine the group of point transformations which this equation admits.

If the equation (27) is invariant under an infinitesimal transformation with the symbol

$$\xi \frac{\partial t}{\partial x} + \eta \frac{\partial t}{\partial y},$$

the general theory of groups gives the condition *

$$\{\eta + y(\eta_y - 2\xi_x - 3\xi_y y')\}(1 + y'^2) - 2yy'\{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}$$
$$+ 2y^2\{-\xi_{yy}y'^3 + (\eta_{yy} - 2\xi_{xy})y'^2 + (2\eta_{xy} - \xi_{xx})y' + \eta_{xx}\} = 0.$$

Equating the coefficients of the various powers of y' to zero, we find the following system of equations for the determination of the unknown functions ξ, η:

$$2y\xi_{yy} + \xi_y = 0, \qquad \eta - y\eta_y + 2y^2\eta_{yy} - 4y^2\xi_{xy} = 0,$$
$$3\xi_y + 2\eta_x + 2y(\xi_{xx} - 2\eta_{xy}) = 0, \qquad \eta + y(\eta_y - 2\xi_x) + 2y^2\eta_{xx} = 0.$$

* LIE-SCHEFFERS, *Differentialgleichungen*, p. 363.

We shall omit the somewhat long but not particularly difficult discussion of these equations and simply state the solution

$$\xi = a_1x^2 + a_2x + a_3, \qquad \eta = 2a_1xy + a_2y. \tag{38}$$

Hence the equation (24) admits three independent infinitesimal transformations with the symbols

$$\frac{\partial f}{\partial x}, \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \qquad x^2\frac{\partial f}{\partial x} + 2xy\frac{\partial f}{\partial y}. \tag{39}$$

The finite equations of the continuous group generated by these infinitesimal transformations may be found by the integration of the simultaneous system

$$\frac{dx_1}{a_1x_1^2 + a_2x_1 + a_3} = \frac{dy_1}{2a_1xy + a_2y} = d\tau,$$

with the initial conditions $\tau = 0$, $x_1 = x$, $y_1 = y$.

When a_1 vanishes, it is found that

$$x_1 = a'x + b', \qquad y_1 = a'y. \tag{40}$$

When a_1 does not vanish, it may be taken equal to unity, and the integrals, after putting $\tau = \log t$, take the form

$$\frac{x_1 + r}{x_1 + s} = t\frac{x + r}{x + s}, \qquad \frac{(x_1 + r)(x_1 + s)}{y_1} = \frac{(x + r)(x + s)}{y} \tag{41}$$

or, explicitly,

$$x_1 = \frac{(st - r)x + rs(t - 1)}{(1 - t)x + s - rt}, \qquad y_1 = \frac{t(s - r)^2y}{\{(1 - t)x + s - rt\}^2}. \tag{41'}$$

Hence, equation (27), *or the system of parabolas with a common directrix, admits the three-parameter continuous groups* H_3 *represented by* (40) *and* (41). *The transformations* (40) *constitute a two-parameter conformal subgroup.*

Consider now the possible representations of a surface S defined by

$$ds^2 = v(du^2 + dv^2).$$

We have already seen that, by the representation

$$R: \qquad x = u, \qquad y = v,$$

the geodesics are pictured by the system (27),

$$\sigma: \qquad 2yy'' = y'^2 + 1.$$

Since σ is transformed into itself by the group H_3, it follows that if R is combined with one of these plane transformations, the resulting representation will still picture the geodesics of S by the system σ. We thus obtain ∞^3 repre-

sentations RH_3, corresponding to the one system σ. Of these, a double infinity, namely RG_2, are conformal.

But there are ∞^4 systems of the second type described in § 2, any one of which may be used instead of the system σ. All these are equivalent to σ with respect to the group G_6 of § 3. To obtain all the systems from σ it is in fact sufficient to employ the four-parameter subgroup

$$G_4: \qquad x_1 = x, \qquad y_1 = e_0 x^2 + e_3 y + e_4 x + e_5.$$

The most general representation of S is therefore obtained by combining G_4 with the representations already found. The result may be indicated by RH_3G_4, and involves seven arbitrary constants. This yields no new conformal representations since G_4 contains no conformal transformations except identity.

If a surface belongs to the class defined by (30) *it may be represented upon the plane so that its geodesics are pictured by parabolas in ∞^7 ways, of which ∞^2 are conformal. The general representation is of the form RH_3G_4, while the conformal is RG_2.*

Let us pass now to the other surfaces belonging to the second type of § 5. Such a surface is geodesically equivalent to those just discussed. The geodesics will still admit a three-parameter group, so that there are ∞^3 representations for the one system σ. From § 6, none of these will be conformal. Our result is:

If a surface belongs to the classes defined by (31), (32), *or* (33), *it is possible to represent it in ∞^7 ways upon a plane so that its geodesics are pictured by parabolas. None of these representations is conformal.*

§ 8. *Additional results.*

We may extend the results obtained by making use of the following

Lemma. If an equation of cubic type (1) is an integral of an equation of the form

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n = 0, \tag{42}$$

where the coefficients are functions of x, y, then $A = 0$, that is, (1) reduces to a quadratic.

To prove this, it is sufficient to observe that, when the values of y'', y''', $\cdots$, $y^{(n)}$ obtained from (1) in terms of y' are substituted in (42), the coefficient of the highest power of y' is a numerical multiple of A^{n-1},

In particular, if the cubic equation is of the form (35), arising in connection with conformal representation, it follows that $L_x = 0$, so that L is a function of y alone. The corresponding linear element (28) then belongs to surfaces applicable on surfaces of revolution.

If a surface can be represented conformally on a plane so that the curves depicting its geodesics satisfy an equation of the form (42) (*of any order but linear in the derivatives*), *then its element is necessarily of the form*

$$ds^2 = f(v)(du^2 + dv^2), \tag{43}$$

so that the surface is applicable on a surface of revolution.

The equation $y''' = 0$, which has been discussed, is of the form (42). We have seen (§ 6) that the corresponding surfaces are the developables and those with the element (30). We shall now show that no new surfaces arise in connection with any equation (42) of the third order.

The equation of the geodesics of (43) is

$$y'' = (1 + y'^2)Y, \tag{44}$$

where Y is a function of y, namely,

$$Y = \frac{f'(y)}{2f(y)}. \tag{44'}$$

By successive differentiation, we find

$$\begin{aligned}
y^{\mathrm{III}} &= y'(1 + y'^2)Y_1, && \text{where } Y_1 = 2Y^2 + Y';\\
y^{\mathrm{IV}} &= (1 + y'^2)(Y_2 y'^2 + Y_3), && \text{where } Y_2 = 3YY_1 + Y_1', \quad Y_3 = YY_1;\\
y^{\mathrm{V}} &= y'(1 + y'^2)(Y_4 y'^2 + Y_5), && \text{where } Y_4 = 4YY_2 + Y_2',\\
& && \quad Y_5 = 2Y(Y_2 + Y_3) + Y_3'; \text{ etc.}
\end{aligned} \tag{45}$$

Taking the equation (42) for $n = 3$, and substituting the values of y' and y''', we have

$$y'(1 + y'^2)Y_1 + (1 + y'^2)YP_1 + y'P_2 + P_3 = 0.$$

Equating the coefficients of powers of y', we obtain a simple set of conditions, with the solution

$$P_1 = 0, \qquad P_2 = 0, \qquad P_3 = 0, \qquad Y_1 = 0.$$

The first three values show that (42) reduces to $y''' = 0$: and the last shows that Y is either a constant or of the form $(2y + c)^{-1}$. This leads, of course, to the surfaces considered in § 6.

The only case in which an equation of the third order of the form (42) *can be satisfied by a system of curves corresponding to the geodesics of a surface in conformal representation, is the case* $y''' = 0$; *the surfaces are then the developables and the class* (30) *described in* § 6.

The examination of the equations $y^{\mathrm{IV}} = 0$, $y^{\mathrm{V}} = 0$, $y^{\mathrm{VI}} = 0$ leads to the same conclusion: no new surfaces are obtained. The same is *probably* true of all equations of the form $y^{(n)} = 0$. There are, however, equations of the fourth order of the form (42) which lead to new solutions. We shall discuss these on another occasion.

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THE DOUBLY PERIODIC SOLUTIONS OF POISSON'S EQUATION IN TWO INDEPENDENT VARIABLES*

BY

MAX MASON

The only doubly periodic solution † of LAPLACE's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is $u = c$, where c is a constant. For if u be such a solution, and v the conjugate potential to u, then $u + iv$ would be a complex analytic function which has a value under a fixed finite limit for all values of x, y. But, as is well known, such a function is necessarily a constant.

It is the object of this paper to investigate, by the use of methods analogous to those of the potential theory, the doubly periodic solutions of POISSON's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{1}$$

where $f(x, y)$ is continuous and periodic in x and in y with the periods a and b respectively.‡ A "doubly periodic GREEN's function," G, will be formed from known functions, and the desired solution of (1) found by quadrature from G and f.

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† A function u will be called a solution of the differential equation within a region Ω, provided that u satisfies the differential equation at every point within Ω. This definition requires the existence of the second derivatives of u at every point in Ω, and therefore the continuity of the first derivatives. By a doubly periodic solution we shall mean a doubly periodic function which is a solution of the equation in the period rectangle, and therefore in the entire plane. Such a function, in particular, has a value less than a fixed finite number for all values of x, y.

‡ In a recent article (Journal de Mathématiques, ser. 5, vol. 10 (1904), p. 445) I have considered by a different method the existence of periodic solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda A(x, y)u = f(x, y),$$

where λ is a parameter.

§ 1. *A doubly periodic Green's function and its law of reciprocity.*

Let $\Re$ denote the real part of the term before which it is written, and consider the function

$$\Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)}; \; z = x + iy, \; \zeta = \xi + i\eta, \; \gamma = a + i\beta,$$

where $\sigma(z)$ is the Sigma function of WEIERSTRASS, formed with the periods a, ib; and ξ, γ are two points in the interior of the period rectangle Ω bounded by the lines $x = 0, y = a, x = 0, y = b$. This function is a solution of Laplace's equation within Ω, except at the points (ξ, η) and (a, β), and has the form

$$\log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + g,$$

where g is a solution of LAPLACE'S equation throughout Ω.

Since for any integers m, n, the function σ obeys the law

$$\sigma(z + ma + inb) = (-1)^{mn+m+n} e^{(m\eta_1 + n\eta_3)(2z+ma+inb)} \sigma(z),$$

where η_1, η_3 are certain complex constants,* we have

$$(2) \qquad \Re \log \frac{\sigma(z+ma+inb-\zeta)}{\sigma(z+ma+inb-\gamma)} = \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} = m\Re 2\eta_1(\zeta-\gamma) - n\Re 2\eta_3(\zeta-\gamma).$$

Define a real function V by the equation

$$V(x, y, \xi, \eta, \alpha, \beta) = \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a} \Re 2\eta_1(\zeta-\gamma) + \frac{y}{b} \Re 2\eta_3(\zeta-\gamma).$$

Since the last two terms are linear in x and y, V has the form

$$V(x, y, \xi, \eta, \alpha, \beta) = \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + S(x, y, \xi, \eta, \alpha, \beta),$$

where S is a known solution of

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0$$

within Ω. Furthermore V is doubly periodic in x, y with the periods a, b, since from (2) the equation results:

*See *e. g.*, BURKHARDT, *Elliptische Functionen*, p. 53.

$$V(x+ma, y+nb, \xi, \eta, a, \beta) = \Re \log \frac{\sigma(z+ma+inb-\zeta)}{\sigma(z+ma+inb-\gamma)}$$

$$+ \frac{x+ma}{a} \Re 2\eta_1(\zeta-\gamma) + \frac{y+nb}{b} \Re 2\eta_3(\zeta-\gamma)$$

$$= \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a} \Re 2\eta_1(\zeta-\gamma) + \frac{y}{b} \Re 2\eta_3(\zeta-\gamma)$$

$$= V(x, y, \xi, \eta, a, \beta)^*.$$

We shall call the function

$$G(x, y, \xi, \eta, a, \beta) = V(x, y, \xi, \eta, a, \beta) - S(a, \beta, \xi, \eta, a, \beta)$$

the doubly periodic Green's function for the periods a, b. This function has the following characteristics:

1°. Except at (ξ, η) and (a, β), G is, within Ω, a solution of the equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial^2 y} = 0.$$

2°. G has the form

$$G = \log \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} - \log \frac{1}{\sqrt{(x-a)^2+(y-\beta)^2}} + R(x, y, \xi, \eta, a, \beta),$$

where R is a known function, which, with respect to the variables x, y, is a solution of LAPLACE's equation within Ω, and which satisfies for all values of ξ, η in Ω, the equation

$$R(a, \beta, \xi, \eta, a, \beta) = 0.$$

3°. G is doubly periodic in x, y with the periods a, b.

The functions G and R obey the following laws of reciprocity:

$$G(x, y, \xi, \eta, \alpha, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} = G(\xi, \eta, x, y, \alpha, \beta)$$

$$+ \log \frac{1}{\sqrt{(\xi-\alpha)^2+(\eta-\beta)^2}},$$

$$R(x, y, \xi, \eta, \alpha, \beta) = R(\xi, \eta, x, y, \alpha, \beta).$$

* In the same manner may be formed a doubly periodic function

$$V(x, y, \xi_1, \eta_1, \cdots, \xi_n, \eta_n) = \sum_{i=1}^{n} c_i \log \sqrt{(x-\xi_i)^2+(y-\eta_i)^2} + S(x, y, \xi_1, \eta_1, \cdots, \xi_n, \eta_n)$$

with any number of logarithmic singularities, where S is a solution of LAPLACE's equation in Ω with respect to the variables x, y, provided that $c_1 + c_2 + \cdots + c_n = 0$.

To prove these laws apply GREEN's theorem,

$$\int\int (v\Delta u - u\Delta v)dxdy = \int \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds,$$

where n is the outward drawn normal, to the region Ω' formed by excluding from Ω the circles $c(\xi, \eta)$, $c(\xi', \eta')$, $c(\alpha, \beta)$ of radius r about the points (ξ, η), (ξ', η'), (α, β) respectively, and choose

$$u = G(x, y, \xi, \eta, \alpha, \beta), \qquad v = G(x, y, \xi', \eta', \alpha, \beta).$$

Since u and v are, within Ω', solutions of LAPLACE's equation the double integral over Ω' is zero. Furthermore, since u and v are doubly periodic in x, y, each assumes equal values at opposite points of the bounding lines of the rectangle Ω, while at these points the normal derivative of each assumes values numerically equal but opposite in sign. Therefore the line integral over the sides of the rectangle Ω is zero, and we have, replacing ds by $rd\theta$,

$$\int_{c(\xi,\eta)} \frac{1}{r} G(x, y, \xi', \eta', \alpha, \beta) rd\theta - \int_{c(\xi',\eta')} \frac{1}{r} G(x, y, \xi, \eta, \alpha, \beta) rd\theta$$

$$+ \int_{c(\alpha,\beta)} \frac{1}{r} \{G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta\} rd\theta + h = 0,$$

where

$$\lim_{r=0} h = 0.$$

But

$$G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta) =$$

$$\log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\xi')^2 + (y-\eta')^2}}$$

$$+ R(x, y, \xi, \eta, \alpha, \beta) - R(x, y, \xi', \eta', \alpha, \beta),$$

and

$$R(\alpha, \beta, \xi, \eta, \alpha, \beta) = 0, \qquad R(\alpha, \beta, \xi', \eta', \alpha, \beta) = 0.$$

We obtain therefore in the limit $r = 0$, by well known methods,

$$G(\xi, \eta, \xi', \eta', \alpha, \beta) - G(\xi', \eta', \xi, \eta, \alpha, \beta)$$

$$+ \log \frac{1}{\sqrt{(\alpha-\xi)^2 + (\beta-\eta)^2}} - \log \frac{1}{\sqrt{(\alpha-\xi')^2 + (\beta-\eta')^2}} = 0,$$

or writing x, y for ξ', η',

$$G(x, y, \xi, \eta, \alpha, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}}$$

$$= G(\xi, \eta, x, y, \alpha, \beta) + \log \frac{1}{\sqrt{(\xi-\alpha)^2 + (\eta-\beta)^2}}.$$

which is the law of reciprocity for G. If we replace G in this equation by its expression from 2° we have immediately,

$$R(x, y, \xi, \eta, \alpha, \beta) = R(\xi, \eta, x, y, \alpha, \beta).$$

Thus R is symmetrical with respect to x, y and ξ, η and is therefore a solution of

$$\frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} = 0$$

within Ω.

§ 2. *The doubly periodic solutions of*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y). \tag{1}$$

Suppose a doubly periodic solution u of equation (1) exists, for the periods a, b, where f is continuous and doubly periodic with the same periods. If a second solution of the same nature existed, the difference of the two would be a doubly periodic solution of LAPLACE's equation, and therefore a constant. It follows that a doubly periodic solution of (1) for the periods a, b is uniquely determined if its value at a fixed point is given.

Apply GREEN's theorem to the period rectangle Ω, choosing for u a doubly periodic solution of (1) and taking $v = 1$. Since the integral over the boundary vanishes, the following equation results:

$$\int_0^b \int_0^a \Delta u \, dx \, dy = \int_0^b \int_0^a f(x, y) \, dx \, dy = 0.$$

This equation is a necessary condition for the existence of a doubly periodic solution of (1). We shall now show that it is also sufficient. Consider the function

$$\begin{aligned} u(\xi, \eta) &= -\frac{1}{2\pi} \int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x, y) \, dx \, dy \\ &= -\frac{1}{2\pi} \int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} f(x, y) \, dx \, dy \\ &\quad + \frac{1}{2\pi} \int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} f(x, y) \, dx \, dy \\ &\quad - \frac{1}{2\pi} \int_0^b \int_0^a R(x, y, \xi, \eta, \alpha, \beta) \, dx \, dy. \end{aligned}$$

Since

$$\frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} = 0,$$

we see at once, from the potential theory, that $u(\xi, \eta)$ is a solution of

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta).$$

Furthermore, putting $\xi = \alpha$, $\eta = \beta$ we have

$$u(\alpha, \beta) = 0,$$

since

$$R(x, y, \alpha, \beta, \alpha, \beta) = R(\alpha, \beta, x, y, \alpha, \beta) = 0.$$

From the law of reciprocity for G we have

$$G(x, y, \xi + ma, \eta + nb) - G(x, y, \xi, \eta, \alpha, \beta)$$

$$= \log \frac{1}{\sqrt{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} - \log \frac{1}{\sqrt{(\xi - \alpha)^2 + (\eta - \beta)^2}},$$

and therefore

$$u(\xi + ma, \eta + nb) - u(\xi, \eta)$$

$$= -\frac{1}{2\pi} \log \sqrt{\frac{(\xi - \alpha)^2 + (\eta - \beta)^2}{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} \int_0^b \int_0^a f(x, y)\, dx dy.$$

Therefore, if

$$\int_0^b \int_0^a f(x, y)\, dx dy = 0,$$

the function u possesses the periods a, b. We have therefore proved the theorems:

The necessary and sufficient condition for the existence of a doubly periodic solution (periods a, b) of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{1}$$

where f is a continuous doubly periodic function with the periods a, b, is that f satisfy the equation

$$\int_0^b \int_0^a f(x, y)\, dx dy = 0.$$

If this condition is satisfied, then the doubly periodic solution of (1), with periods a, b, which assumes the value C at $x = \alpha$, $y = \beta$ is uniquely determined, and is given by the formula:

$$u(\xi, \eta) = -\frac{1}{2\pi} \int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x, y)\, dx dy + C,$$

where G is a known function, expressible in terms of Sigma functions.

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DEFINITION IN TERMS OF ORDER ALONE IN THE LINEAR CONTINUUM AND IN WELL-ORDERED SETS*

BY

OSWALD VEBLEN

First definition of the continuum.

A linear continuum is a set of elements $\{P\}$ which we may call points, subject to a relation $\prec$ which we may read *precedes*, governed by the following conditions, A to E:

A. General Postulates of Order.

1. *$\{P\}$ contains at least two elements.*
2. *If P_1* and *P_2 are distinct elements of $\{P\}$ then either $P_1 \prec P_2$ or $P_2 \prec P_1$.*
3. *If $P_1 \prec P_2$ then P_1 is distinct from P_2.*†
4. *If $P_1 \prec P_2$ and $P_2 \prec P_3$, then $P_1 \prec P_3$.*†

B. Dedekind Postulate of Closure.‡

If $\{P\}$ consists entirely of two infinite subsets, $[P']$ and *$[P'']$ such that $P' \prec P''$, then there is either a P'_0* of *$[P']$ such that $P' \prec P'_0$ whenever $P' \neq P'_0$ or a P''_0 of $[P'']$ such that $P''_0 \prec P''$ whenever $P'' \neq P''_0$.*

C. Pseudo-archimedean postulate.

1. *If $\{P\}$ is an infinite set then there exists a subset $[P_\nu]$, such that $P_\nu \prec P_{\nu+1} (\nu = 1, 2, 3, \cdots)$ and if P is any element of $\{P\}$ for which $P_1 \prec P$, then there exists a ν such that $P \prec P_\nu$.*

* Presented to the Society December 30, 1904, uuder the title, *Non-Metrical Definition of the Linear Continuum.* Received for publication January 26, 1905.

† The propositions A_3 and A_4 say that $\prec$ is "non-reflexive" and "transitive." It follows immediately that it is "non-symmetric," for if $P_1 \prec P_2$ and $P_2 \prec P_1$ could occur simultaneously, 4 would lead to $P_1 \prec P_1$, contrary to 3.

‡ Usually called a postulate of continuity; but a set would hardly be called continuous unless it also satisfied condition D.

2. *If* $\{P\}$ *is an infinite set then there exists a subset* $[P'_\nu]$, *such that* $P'_{\nu+1} \prec P'_\nu (\nu = 1, 2, 3, \cdots)$ *and if* P *is any element of* $\{P\}$ *for which* $P \prec P'_1$, *then there exists a* ν *such that* $P'_\nu \prec P$.

D. Postulate of density.

If P_1 *and* P_3 *are any two elements of* $\{P\}$ *such that* $P_1 \prec P_3$ *then there exists an element* P_2 *of* $\{P\}$ *such that* $P_1 \prec P_2$ *and* $P_2 \prec P_3$.

Definition. An element P_2 such that $P_1 \prec P_2 \prec P_3$ is said to lie *between* P_1 and P_3. If P_1 and P_3 have an element between them the set of all such elements is called the *segment* P_1P_3. Thus the segment P_1P_3 does not contain either P_1 or P_3 which are called its *end-points*. A segment *plus* its end-points is called an interval. One segment is said to contain another if it contains all its elements.

E. Postulate of uniformity.

For every element P *of a set* $\{P\}$ *which satisfies conditions* A, B, C, D, *and for every* $\nu (\nu = 1, 2, 3, \cdots)$ *there exists a segment* * $\sigma_{\nu P}$ *such that the set of segments* $[\sigma_{\nu P}]$ *has the following properties* †:

1. *For a fixed* P, $\sigma_{\nu P}$ *contains* $\sigma_{\nu+1 P}$.

2. *For a fixed* P, P *lies on every* $\sigma_{\nu P}$ *and is the only such element.*

3. *For every segment* τ, *there exists a* ν, ν_τ, *such that for no* P *does* $\sigma_{\nu_\tau P}$ *contain* τ.

Relations of the above to other definitions.

Of the above postulates or items of the definition, those grouped under the heading A constitute the usual definition of *order* which probably received its first explicit formulation in the hands of G. Cantor. They can easily be shown equivalent to the postulates given by other writers ‡ in terms of the relations of "precedence" or "betweenness" or "separation." The proof of this equivalence and the deduction of the elementary theorems of order are here omitted since they are covered fairly well by the literature cited in the footnote above.

From conditions A and B follow the theorems that every bounded set has a

* The notation $\sigma_{\nu P}$ indicates that $\sigma_{\nu P}$ is a function both of ν and of P.

† A metrical special case of a set $[\sigma_{\nu P}]$ is obtained by letting $\sigma_{\nu P}$ be a segment of length $1/\nu$ whose middle point is P. It is evident that as ν increases the segment $\sigma_{\nu P}$ closes down on the points P in the *uniform* manner described in number 3 above. The theorem of uniform continuity involves consideration of a set like $[\sigma_{\nu P}]$.

‡ Cf. footnote on page 88 of this volume; and also E. V. Huntington, *A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups*, pages 17–41 of this volume.

least upper and greatest lower bound,* that every infinite bounded set has at least one limiting element, and also the important.

HEINE-BOREL THEOREM. *If every element of an interval P_1P_2 belongs to at least one segment σ of a set of segments $[\sigma]$, then there exists a finite subset of $[\sigma]$, $\sigma_1, \sigma_2, \cdots, \sigma_n$, such that every element of P_1P_2 belongs to at least one of the segments $\sigma_1, \sigma_2, \cdots, \sigma_n$.* This theorem, as well as a generalization of it, is proved in the Bulletin of this society, vol. 10, p. 436, by a method which applies to our hypotheses. We may note in passing, what seems to be a new result, that, since conditions A and B are satisfied by any well-ordered set in the sense of CANTOR, *the Heine-Borel theorem is a true theorem of any well-ordered set.*

Conditions A and B are sufficient for a large portion of the theory of functions of a real variable. Condition D provides that every element of $\{P\}$ shall be a limiting element. The three conditions A, B, D are common to all current definitions of the continuum in so far as they involve the notion of order.

At this point there have been three essentially different modes of advance. The first, which we may call analytical, introduces a symbol for the combining of any two elements to produce a third, i. e., regards the elements as numbers.† The second, a geometrical method, brings in the notion of congruence of segments. Either of these schemes provides a relation between segments at different positions in the continuum, thus affording a basis for theorems of *uniformity*, e. g., uniform continuity. The third method, that of G. CANTOR, provides for this uniformity by postulating the presence of an everywhere dense subset ordinally similar to the ordinary rational numbers. CANTOR'S definition‡ is equivalent to A, B, D, a condition that the continuum shall have two endpoints, and the following:

F. $\{P\}$ *contains a subset* $\{P\}$ *of cardinal* $\aleph_0$ *(i. e., an enumerable subset) such that between every two elements of* $\{P\}$, *there is an element of* $\{P\}$.

CANTOR'S definition is simpler than the other two in that relations of congruence or operations of addition and multiplication do not appear among his undefined symbols. On the other hand he introduces the notion of infinite cardinal number. The definitions furnished by our conditions A, B, C, D, E are stated entirely in terms of the relation $\prec$ and the general logical symbols that appear in every mathematical discussion. While the language in which our

* An *upper bound* of a set $[x]$ is an element, B, such that $x \prec B$ for every $x \neq B$. A *least* upper bound is an upper bound B_0 such that if B is any other upper bound of $[x]$, $B_0 \prec B$. A *limiting element* of a set $[x]$ is an element A such that every segment which contains A contains an element of $[x]$ different from A.

† Cf. E. V. HUNTINGTON, loc. cit.

‡ G. CANTOR, *Zur Begründung der transfiniten Mengenlehre*, I, Mathematische Annalen, vol. 46 (1895), p. 510.

condition E is stated may be a little more complicated than that used in the other definitions, it is often more convenient of application.*

CANTOR has proved, loc. cit., p. 510, that any two sets satisfying his conditions are in a one-to-one correspondence preserving order relations, i. e., that his definition is categorical.† Therefore, to show the same for our definition, we need only to show that F is a consequence of our postulates. Among other results of this theorem it may be remarked that, since a continuum according to our definition is in one-to-one reciprocal correspondence with the continuum defined by the usual scheme of deriving irrational numbers from rationals, we have (by mediation of the correspondence) a definition of addition and multiplication in terms of the relation $\prec$.‡

Proof of condition F.

Let $P_1 P_1'$ be any interval of the continuum and let $[\sigma'_{\nu_1}]$ be that subset of the set $[\sigma_{\nu_1}]$ referred to in postulate E for which the subscript point P lies on the interval $P_1 P_1'$. The subset of $[\sigma'_{\nu_1}]$ obtained by fixing a value of ν_1 is an infinite set such that every point P of $P_1 P_1'$ lies on one segment σ'_{ν_1}. By the HEINE-BOREL theorem, there is a finite subset of this infinite set having the same property. The end-points of the segments of this set are a finite set of points

$$P_\nu^1 P_\nu^2 \cdots P_\nu^m{}_\nu.$$

The set of all such points $[P_\nu^m{}_\nu]$, $\nu = 1, 2, \cdots$, is evidently enumerable and by the third item in condition E, has a point between any two points of the interval $P_1 P_1'$.

Consider now the sets of points $[P_\nu]$ and $[P_\nu']$ of postulate C. There evidently exists an integer n such that

$$P_n' \prec P_n.$$

* Compare, for example, *Theory of plane curves in non-metrical analysis situs*, these Transactions, vol. 6 (1905), p. 83.

† The word "categorical" implies that between any two classes $\{P\}$ and $\{Q\}$ satisfying the conditions named, there is a one-to-one correspondence such that if $P_1 \prec P_2$ then for the corresponding elements of $\{Q\}$, $Q_1 \prec Q_2$. It therefore also implies that any proposition stated in terms of the relation $\prec$ either is true for all classes satisfying our conditions or is false for all such classes. *Cf.* vol. 5, p. 346 of these Transactions and also some remarks of E. V. HUNTINGTON in his paper on Complex Algebra in the present number.

‡ An interesting situation is obtained by introducing a postulate of uniformity among the hypotheses of plane analysis situs (cf. p. 84 of this volume). If the postulate is applied to the straight line, the line is necessarily a continuum but it is not obvious that other curves are. If it is applied to the plane, the segments $\sigma_{\nu P}$ in this case being triangular regions, all continuous curves are continua, but it is not obvious that there is a one-to-one reciprocal correspondence between the plane and a set of number-pairs. There seems thus to be possible a still more general formulation of the RIEMANN-HELMHOLZ-LIE problem of the foundations of geometry than that obtained by HILBERT.

The enumerable set of intervals $P'_{n+k}P_{n+k}$, $k = 1, 2, \cdots$, is evidently such that for any point P, there is a value of k such that P lies on $P'_{n+k}P_{n+k}$. On each of these segments there is, by the paragraph above, an enumerable everywhere dense set of points. The points of the enumerable set, so obtained, of enumerable everywhere dense sets of points constitute an enumerable and everywhere dense set as required by the conclusion of F.

Independence of the postulates.

The independence of the items of conditions A is proved by the examples, A_1, A_2, A_3, A_4 given on pages 89–90 of this volume of the Transactions. The independence of condition B is proved by example B_2 and of condition D by example B_1 on the same page.

To prove the independence of condition C_1, we observe that a well-ordered set of type† Ω (the set of all well-ordered sets of cardinal number $\aleph_0$) does not satisfy condition C. For if a sequence such as $[P_\nu]$ should exist the set of all elements preceding each P_ν would be enumerable and hence Ω itself would be enumerable, contrary to CANTOR'S theorem. If each element of this set and its next following element are connected by a set similar to the continuum of real numbers greater than zero and less than one, the resulting set satisfies conditions A, B, D, E (E vacuously), but not C_1. The independence of C_2 is proved by employing $^*\Omega$ instead of Ω as above.

To prove the independence of E, consider the set of all pairs of real numbers (x, y), $-\infty < x < +\infty$, $0 \leqq y \leqq 1$, the relation $\prec$ being defined so that

$$(x_1y_1) \prec (x_2y_2) \text{ when } x_2 \neq x_1 \text{ if } x_1 \prec x_2,$$

$$(x_1y_1) \prec (x_2y_2) \text{ when } x_2 = x_1 \text{ if } y_1 \prec y_2.$$

That A, C, D are satisfied is evident; B is also seen easily if the DEDEKIND postulate is applied first to the variable x and then to y. If E were satisfied, F, which is a consequence of it, would also be satisfied; but an everywhere dense subset would have to include a point for every value of x, whereas $[x]$ is not enumerable.

Postulates for well-ordered sets.

A set of independent postulates defining the general notion, well-ordered set, is obtained by taking together, with conditions A and B, the following:

W_1. *The set* $\{P\}$ *contains an element* P_0 *such that for no element* P' *does* $P' \prec P_0$.

P_0 is called the *first* element.

† See CANTOR, loc. cit., p. 207.

W_2. *If P_1 is any element of $\{P\}$ for which there are elements P', $P_1 \prec P'$, $\{P\}$ contains an element P_2 such that $P_1 \prec P_2$ and such that for no element P'' do there exist both of the relations,*

$$P_1 \prec P'' \text{ and } P'' \prec P_2,$$

P_2 is called the element *next following* P_1, and P_1 the element next preceding P_2.

There is no difficulty in seeing that conditions W_1, W_2, and B are equivalent to the conditions of CANTOR'S definition of a well-ordered set.* CANTOR defines particular classes of well-ordered sets by means of their cardinal numbers. This can also be done purely ordinally. For example, a finite set is one which satisfies conditions A, B, W_1, W_2, together with W_3 and W_4.

W_3. *Every element of $\{P\}$ except P_0 has a next preceding element.*

W_4. *The set $\{P\}$ contains an element $\overline{P}$ such that for no element P' does $\overline{P} \prec P'$.*

$\overline{P}$ is called a *last* element.

A set of type ω is determined by conditions A, B, W_1, W_2, W_3, together with W_ω, which is the negative of W_4.

W_ω. *If P is any element of $\{P\}$, $\{P\}$ contains an element P' such that $P \prec P'$.*

A set of the type associated with the number-class $Z(\aleph_0)$, i. e., the type of set which CANTOR defines by prescribing the cardinal $\aleph_0$, is given by conditions A, B, W_1, W_2, together with $W_{3'}$, and $W_{4'}$:

$W_{3'}$. *Every element of $\{P\}$ except P_0 has a next preceding element or is the least upper bound of a subset of type ω (as just defined).*

$W_{4'}$. *$\{P\}$ either contains a last element or a subset $\{P_\nu\}$ of type ω such that for every P, there exists a ν_P for which $P \prec P_{\nu_P}$.*

The proof may readily be given that a set satisfying these conditions is enumerable. To define the type Ω we take A, B, W_1, W_2, $W_{3'}$ together with W_Ω which is the exact negative of $W_{4'}$, i. e., with "neither" substituted for "either" and "nor" for "or." To define the type of sets associated with the number-class $Z(\aleph_1)$, we take A, B, W_1, W_2 together with $W_{3''}$ and $W_{4''}$.

$W_{3''}$. *Every element of $\{P\}$ except P_0 has a next preceding element or is the least upper bound of a subset of type ω or Ω.*

$W_{4''}$. *$\{P\}$ either contains a last element or a subset $[Q]$ of type ω or Ω such that for every P there exists a Q_P for which $P \prec Q_P$.*

This scheme of definition is evidently adequate to determine any class of well-ordered sets.

*CANTOR, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematische Annalen, vol. 49 (1898), p. 207.

Modification of the definition of the continuum.

By the following analysis, the postulate of uniformity can be made formally a little weaker. Adopting the language of CANTOR, let us denote by the words *fundamental sequence* a subset of $\{P\}$ of type ω — that is, a subset $\{P_\kappa\}$, $\kappa = 1, 2, 3, \cdots$, where $P_\kappa \prec P_{\kappa+1}$. Any element which is the least upper bound of a fundamental sequence is a *principal element.* Postulate E may now be replaced by the following two, E' and E''.

E'. *Let* $[P']$ *be the set of all principal elements of* $\{P\}$; *if any such exist, there exists a set of sets of points* $[\{P_\nu\}_{P'}]$ *such that:*

1. *If* P' *is any point of* $[P']$, *the set* $\{P_\nu\}_{P'}$ *has* P' *as least upper bound and is such that* $P_\nu \prec P_{\nu+1}$.

2. *If* P'' *and* $P'''(P'' \prec P''')$ *are any two points of* $[P']$, *there is a value* ν_1 *of* ν *such that for no point* P' *do there exist the relations*

$$P_{\nu_1} \prec P'', \; P''' \prec P',$$

if P_{ν_1} *belongs to the set* $\{P_\nu\}_{P'}$

E''. *Every element of* $\{P\}$ *is a principal element.*

The proof that condition F is a consequence of A, B, C, D, E needs to be modified when E is replaced by E' and E''. By E'', the set $[P']$ of E' is identical with $\{P\}$. For a fixed value of ν, the segments $P_\nu P = \sigma_{\nu P}$, where P_ν belongs to the set $\{P_\nu\}_P$, constitute a set to each segment of which, every point of $\{P\}$ is either an interior or an end point.

Fixing attention on a particular interval $P_1 P_1'$ of the continuum, for a fixed value of ν its points fall into two classes, those which are interior points of segments $\sigma_{\nu P}$ (for P on $P_1 P_1'$) and those which are end points of such segments without being interior to any. Call this latter set $[\overline{P}^\nu]$.

Defining the relation $\succ$ so that $P_1 \succ P_2$ if and only if $P_2 \prec P_1$, we easily see that with respect to the sense $\succ$, $[\overline{P}^\nu]$ is a well ordered set of which P_1' is the first element. But in view of condition E'' it satisfies conditions W_3 and W_4 as well as A, B, W_1, W_2. $[\overline{P}^\nu]$ is therefore enumerable, and its points are the upper end points in the sense $\prec$ of an enumerable set of intervals which include all points of $P_1 P_1'$ as interior or end points. By applying the HEINE-BOREL theorem to these intervals separately, we obtain, instead of a finite set as when using condition E, a numerably infinite set

$$P_\nu^1, P_\nu^2, \cdots P_\nu^{m_\nu} \cdots.$$

The set $[P_\nu^{m_\nu}]$, $\nu = 1, 2, 3 \cdots$ is enumerable and everywhere dense on $P_1 P_1'$, and the rest of the argument proceeds as in the former case.

ON THE STRUCTURE OF HYPERCOMPLEX NUMBER SYSTEMS*

BY

SAUL EPSTEEN AND J. H. MACLAGAN-WEDDERBURN†

In this paper theorems regarding linear associative algebras are enunciated which are analogous to the JORDAN-HÖLDER theorems concerning the quotient groups and indices of composition in the theory of finite groups (cf. BURNSIDE, *Theory of Groups*, chapter 7); and of the corresponding VESSIOT-ENGEL theorems in the theory of continuous groups (Annales de l'École Normale Supérieure, vol. 9 (1892), p. 203; LIE-ENGEL, *Transformationsgruppen*, vol. 3, pp. 704, 765).

The methods used throughout are rational and hence the results apply to hypercomplex number systems in which the coefficients are restricted to be marks of a given field, finite or infinite.

§1.

We consider the linear associative algebra

$$E \equiv e_1 \cdots e_m e_{m+1} \cdots e_r e_{r+1} \cdots e_n, \tag{1}$$

where

$$e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3} \qquad (i = 1, \cdots, n). \tag{2}$$

From the associativity property $(e_{i_1} e_{i_2}) e_{i_3} = e_{i_1} (e_{i_2} e_{i_3})$ and the linear independence of the units are deduced the relations

$$\sum_{i_3} (\gamma_{i_1 i_2 i_3} \gamma_{i_3 i_4 i_5} - \gamma_{i_1 i_3 i_5} \gamma_{i_2 i_4 i_3}) = 0 \quad (i_1, i_2, i_4, i_5 = 1, \cdots, n). \tag{3}$$

Addition of complexes. If $X_1, \cdots, X_m$ are any m independent numbers of the system E, all numbers linearly dependent on the $X_1, \cdots, X_m$ and only on these numbers are said to form a complex of order m. We may write $E_1 = X_1 \cdots X_m$. If, similarly, $E_2 = X'_1 \cdots X'_{m'}$, the complex formed of all members linearly dependent on $X_1, \cdots, X_m, X'_1, \cdots, X'_{m'}$ is called the sum of E_1 and E_2. The order of $E_1 + E_2$ is not greater than the sum of the orders of E_1 and E_2 but may be less; thus if $E_1 = e_1 \cdots e_m \cdots e_r$, $E_2 = e_{m+1} \cdots e_r \cdots e_{m'}$ the sum $E_1 + E_2$ is of order m'. According to this convention, $E + E_1 = E$, $E + E_2 = E$, $E + E = E$.

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*Multiplication of complexes.** When E_1 and E_2 are defined as above, the product $E_1 E_2$ is defined as the complex of all numbers linearly dependent on $X_p X'_q$ ($p = 1, \cdots, m$; $q = 1, \cdots, m'$). It is easily shown from the definition of multiplication of complexes that the process is associative. Thus if we consider the system

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

and let $E_1 \equiv e_1$, $E_2 \equiv e_2$, then $E_1 E_2 = e_1 e_2 = e_2$. It can easily be seen that if the main system E contains a modulus (say e_1) then $EE(= E^2) = E$; for, the modulus (unit) e_1 multiplied by $e_1 \cdots e_n$ on the right gives $e_1 \cdots e_n$ and therefore E^2 contains all the units of E. If E does not contain a modulus it may happen that in E^2 some of the units $e_1, \cdots, e_n$ do not appear and therefore $E^2 \preceq E$. The condition that E be a system, closed under multiplication, is $E^2 \preceq E$.

Semireducibility. The algebra E is said to be semireducible† when by a proper choice of units the following conditions are fulfilled:

$$X \cdot {}_2X = {}_2X' \quad \left.\right\} \text{ for every } X, {}_2X, \tag{4}$$

$$ {}_2X \cdot X = {}_2X'' \quad \left.\right\} \tag{5}$$

the general number of E being

$$X = {}_1X + {}_2X' = \sum_{j=1}^{m} x_j e_j + \sum_{k=m+1}^{n} x_k e_k.$$

It follows at once that the complex $E_2 = e_{m+1} \cdots e_n$ is closed under multiplication. The conditions (4) and (5) may be written

$$E \cdot E_2 \preceq E_2, \tag{4'}$$

$$E_2 E \preceq E_2, \tag{5'}$$

and, therefore, of course, $E_2 E_2 = E_2^2 \preceq E_2$. Under these conditions the subalgebra E_2 is said to be invariant in E.‡

THEOREM. *E_2 cannot contain the modulus of E among its numbers.* For, from (4) and (5) it is impossible that any number ${}_2I$ of E_2 shall have the property that ${}_2I \cdot X = X \cdot {}_2I = X$ for every number X of E.

Accompanying system,§ complementary system. If

$$E = E_1 + E_2 \equiv e_1 \cdots e_m e_{m+1} \cdots e_n,$$

* FROBENIUS, Berliner Sitzungsberichte, 1895, p. 164.

† Transactions, vol. 4 (1903), p. 440.

‡ CARTAN, Annales de Toulouse, vol. 12 (1898).

§ MOLIEN'S *Begleitendes System*, Mathematische Annalen, vol. 41 (1893), p. 93.

we may write, in well known notation, $E = E_1 (\mathrm{mod}\ E_2)$, $E = E_2 (\mathrm{mod}\ E_1)$; similarly, if $X = {}_1X + {}_2X$, where X belongs to E, ${}_1X$ to E_1, ${}_2X$ to E_2, we may write $X = {}_1X (\mathrm{mod}\ E_2)$ or $X = {}_2X (\mathrm{mod}\ E_1)$. Regarding as equal* any two numbers of E which are equal, modulo E_2, we get, when E_2 is an invariant subsystem, a hypercomplex number system K which is said to *accompany* E and to be *complementary to the invariant subsystem* E_2 *with respect to* E. It is evidently sufficient to show that the system so defined is associative. If ${}_1X_\alpha$, ${}_1X_\beta$, ${}_1X_\gamma$ are three numbers of E_1 and if

$$ {}_1X_\alpha \cdot {}_1X_\beta = X_{\alpha\beta} = {}_1X_{\alpha\beta} + {}_2X_{\alpha\beta}, \qquad {}_1X_\beta \cdot {}_1X_\gamma = X_{\beta\gamma} = {}_1X_{\beta\gamma} + {}_2X_{\beta\gamma}, $$

then

$$ ({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma = {}_1X_{\alpha\beta} \cdot {}_1X_\gamma + {}_2X_{\alpha\beta} \cdot {}_1X_\gamma = {}_1X_{(\alpha\beta)\gamma} + {}_2X' \quad . $$

if $E_2 E_1 \gtrless E_2$. Similarly

$$ {}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) = {}_1X_{\alpha(\beta\gamma)} + {}_2X''. $$

Now by the associative law and the linear independence of the units we have

$$ {}_1X_{(\alpha\beta)\gamma} \equiv {}_1X_{\alpha(\beta\gamma)}, $$

and therefore

$$ ({}_1X_\alpha \cdot {}_1X_\beta) \cdot {}_1X_\gamma \equiv {}_1X_\alpha \cdot ({}_1X_\beta \cdot {}_1X_\gamma) (\mathrm{mod}\ E_2). $$

The above may also be deduced from a consideration of the $\gamma_{i_1 i_2 i_3}$'s. Using (3) it is seen that

$$ (3_j) \qquad \sum_{i_3=1}^{n} (\gamma_{j_1 j_2 i_3} \gamma_{i_3 j_4 j_5} - \gamma_{j_1 i_3 j_5} \gamma_{j_2 j_4 i_3}) = 0 \qquad (j_1, j_2, j_4, j_5 = 1, \cdots, m). $$

Now if

$$ E_2 E_1 \gtrless E_2, \qquad E_1 E_2 \gtrless E_2, $$

or,

$$ {}_2X \cdot {}_1X = {}_2X', \qquad {}_1X \cdot {}_2X = {}_2X'', $$

that is to say, if every $\gamma_{kjj} = 0$ and every $\gamma_{jkj} = 0$, we may write (3_j) in the form

$$ (3'_j) \qquad \sum_{j_3=1}^{m} (\gamma_{j_1 j_2 j_3} \gamma_{j_3 j_4 j_5} - \gamma_{j_1 j_3 j_5} \gamma_{j_2 j_4 j_3}) = 0.\dagger $$

§ 2.

By $E_1 \frown E_2$ is meant the common complex of E_1 and E_2. Thus if in (1), $E_1 \equiv e_1 \cdots e_m \cdots e_r$, $E_2 \equiv e_{m+1} \cdots e_r \cdots e_n$, then $E_1 \frown E_2 \equiv e_{m+1} \cdots e_r$. $E_1 \frown E_2 = 0$ states that E_1 and E_2 have no numbers in common.

* FROBENIUS, l. c., p. 634.

† According to MOLIEN, loc. cit., p. 92, it is necessary that $EE_2 < E_2$, $E_2 E \gtrless E_2$. Evidently, from the text, we can have an accompanying system with the milder conditions $E_2 E_1 \gtrless E_2$, $E_1 E_2 \gtrless E_2$.

It may be remarked at this point that the γ_{iij} are the structure constants of K.

Reducibility. According to PEIRCE* an algebra E is said to be mixed (or reducible) if new units can be introduced such that $E = E_1 + E_2$, where $E_1^2 \equiv E_1$, $E_2^2 \equiv E_2$, $E_1 E_2 \equiv E_1 \frown E_2$, $E_2 E_1 \equiv E \frown E_2$.†

THEOREM I. *If E_1 is a maximal invariant subalgebra of E and if there exists a second invariant subalgebra E_1' of E, then either E is reducible or else E_1' is a subalgebra E_1.*

Since E_1 and E_1' are invariant we have (a) $EE_1 \equiv E_1$, (b) $E_1 E_1' \equiv E_1$, (c) $EE_1' \equiv E_1'$, (d) $E_1' E \equiv E_1'$. We have now that $E_1 + E_1'$ is an invariant subalgebra of E for $E(E_1 + E_1') \equiv E_1 + E_1'$, $(E_1 + E_1')E \equiv E_1 + E_1'$ and since E_1 is is maximal by hypothesis $E_1 + E_1' = E$ or else $E_1 + E_1' = E_1$. If $E_1 + E_1' = E$ the algebra is easily seen to be reducible. First, $E_1^2 \equiv E_1$ by (a), $E_1'^2 \equiv E_1'$ by (c), $E_1 E_1' \equiv E_1$ by (b) and $E_1 E_1' \equiv E_1'$ by (c) and therefore $E_1 E_1' \equiv E_1 \frown E_1'$. Similarly by (b) and (d), $E_1' E_1 = E_1 \frown E_1'$. In the second case, where $E_1 + E_1' = E_1$, it is evident that E_1' is a subalgebra of E_1.

THEOREM II. *If E_1 and E_1' are maximal invariant subalgebras of E, and if $E_1 \frown E_1' = F \neq 0$, then F is a maximal invariant subalgebra of both E_1 and E_1'.*

By theorem I, $E = E_1 + E_1'$. Let

$$E = E_0 + E_1 \qquad (E_0 \frown E_1 = 0),$$
$$= E_0' + E_1' \qquad (E_0' \frown E_1' = 0).$$

Evidently then $E = E_0 + E_0' + F$. Because the products $E_1 F$, FE_1, $E_1' F$, FE_1' must be contained in E_1 and in E_1' it follows that

$$E_1 F \equiv F, \qquad FE_1 \equiv F, \qquad E_1' F \equiv F, \qquad FE_1' \equiv F,$$

and therefore that F is invariant in both E_1 and E_1'.

Suppose now that F is not maximal in E_1 and let $G + F$ (where $G \frown F = 0$) be a maximal invariant subsystem of E_1, i. e.,

$$E_1(G + F) \equiv G + F, \qquad (G + F)E_1 \equiv G + F.$$

Consider the complex $G + E_1'$. Clearly $G \frown E_1' = 0$ and

$$GE_1' \equiv F, \qquad E_1' G \equiv F,$$

since the products must both lie in E_1 and E_1'. Therefore, since

$$E_1 G \equiv G + F, \qquad E_1 E_1' \equiv F, \qquad E_1' G \equiv F, \qquad E_1'^2 \equiv E_1',$$

we have

$$E(G + E_1') = (E_1 + E_1')(G + E_1') \equiv G = E_1'.$$

It is shown in a similar manner that $(G + E_1')E \equiv G + E_1'$. Thus, if F is not a maximal invariant subalgebra of E_1, E_1' will not be a maximal invariant

* American Journal, vol. 4 (1881), p. 100.

† In SCHEFFERS's definition, Mathematische Annalen, vol. 39 (1891), p. 317, $E_1 \frown E_2 = 0$.

subalgebra of E (since $G + E_1'$ has been proved invariant and $G \frown E_1' = 0$), which is contrary to the hypothesis. It is shown in exactly the same way that F is maximal in E_1'.

THEOREM III. *Let $E, E_1, E_2, \cdots$ be a normal series of subalgebras of E (i. e., E_r is a maximal invariant subalgebra of E_{r-1}, $E_0 = E$) and let $K_1, K_2, \cdots$ be a series of complementary algebras, such that K_r accompanies E_{r-1} and is complementary to E_r. Under these assumptions the series $K_1, K_2, \cdots$ is, apart from the order, independent of the choice of the series $E, E_1, E_2, \cdots$. In other words, if $E, E_1', E_2', \cdots$ is any other normal series, the complementary series of algebras $K_1', K_2', \cdots$ which it defines is the same as the series $K_1, K_2, \cdots$, apart from the sequence.*

According to the demonstration of theorem 1, we have (if $E_1 \neq E_1'$)

$$E = E_1 + E_1'. \tag{6}$$

Let $E_1 \frown E_1' = F_1$, then

$$E_1 = F_1 + D_1 \qquad (F_1 \frown D_1 = 0), \tag{7}$$

$$E_1' = F_1 + D_1' \qquad (F_1 \frown D_1' = 0),$$

and it is evident that

$$E = F_1 + D_1 + D_1'. \tag{8}$$

By theorem 2, F_1 is a maximal invariant subalgebra of E_1 and E_1'.

If F_1, F_2, F_3 is a normal series of F_1, then $E, E_1, F_1, F_2, \cdots$ and $E, E_1', F_1, F_2, \cdots$ are two new normal series of E. We will now have to prove that the proposition is true for these two series. For this purpose it is merely necessary to prove that the complementary algebras defined by E, E_1, F_1 and E, E_1', F_1 are the same. If

$$E \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

$$E_1 \equiv e_1 \cdots e_{d_1} e_{d_1+1} \cdots e_t,$$

$$E_1' \equiv e_{d_1+1} \cdots e_t e_{t+1} \cdots e_n,$$

then

$$E_1 \frown E_1' = F_1 \equiv e_{d_1+1} \cdots e_t,$$

$$D_1 \equiv e_1 \cdots e_{d_1}, \qquad D_1' = e_{t+1} \cdots e_n.$$

Now the complementary algebra of $E_1 = F_1 + D_1$ with respect to E is defined by the $(n-t)^3$ characteristic constants $\gamma_{p_1 p_2 p_3}$ where

$$\sum_{p_3} (\gamma_{p_1 p_2 p_3} \gamma_{p_3 p_4 p_5} - \gamma_{p_1 p_3 p_5} \gamma_{p_2 p_4 p_3}) = 0 \quad (p = t+1, t+2, \cdots, n). \tag{9}$$

Furthermore, the complementary algebra of F_1 with respect to $E_1 (E_1 = F_1 + D_1)$ is defined by the d_1^3 characteristic constants $\gamma_{q_1 q_2 q_3}$ where

$$\sum_{q_3} (\gamma_{q_1 q_2 q_3} \gamma_{q_3 q_4 q_5} - \gamma_{q_1 q_3 q_5} \gamma_{q_2 q_4 q_3}) = 0 \qquad (q = 1, \cdots, d_1). \tag{10}$$

Now it is seen in the same manner that the complementary algebra of E_1' with respect to E is defined by (10) and the complementary algebra of F_1 with repect to E_1' is defined by (9), and therefore the normal complementary series defined by E, E_1, F_1 is identical, apart from the sequence, with the normal complementary series defined by E, E_1', F_1.

It is now necessary to show that the normal series $E, E_1, E_2 \cdots, E, E_1, F_1, F_2, \cdots$ define the same complementary series, and similarly for $E, E_1', E_2', \cdots, E, E_1', F_1, F_2, \cdots$. This amounts to proving the theorem for the algebras E_1 and E_1', which involves a finite number of repetitions of the above proof.

§ 3.

Let $E, E_1, E_2, \cdots$ be a normal series of E, let a_s be the order (number of units) of E_s and let l_s be the difference between a_{s-1} and the maximal order of a subalgebra * of E_{s-1} which contains $E_s (E_0 = E)$.

THEOREM IV. *The numbers $l_1, l_2, \cdots$, apart from their sequence, are independent of the choice of the normal series.*

This theorem is an immediate consequence of theorem 3, the integer l_s being regarded as the difference between the order of the complementary algebra A_s of E_s with respect to E_{s-1} and the order of a maximal subalgebra of $K_s (s = 1, 2, \cdots)$.

Similarly it follows also *that the series of integers $k_s = a_{s-1} - a_s$ is independent of the choice of the normal series.*

Simple algebras. An algebra which contains no invariant subalgebra is said to be simple.

THEOREM V. *The complementary algebras $K_1, K_2, \cdots$ defined by the normal series $E, E_1, E_2 \cdots$ are all simple.*†

In order to prove this, consider the algebra $A = B + A_1$ (where $B \frown A_1 = 0$) wherein A_1 is a maximal invariant subalgebra, and let K be the complementary system of A_1 with respect to A. Suppose now that K has an invariant subalgebra K_1, $K = K' + K_1$ (where $K' \frown K_1 = 0$). This says that

$$KK_1 \precsim K_1, \qquad K_1 K \precsim K_1. \tag{11}$$

From the correspondence between K and B‡ it is seen that we can write $B = B' + B_1$ (where $B' \frown B_1 = 0$) where, by means of (11),

$$BB_1 \precsim B_1 + A_1, \qquad B_1 B \precsim B_1 + A_1. \tag{12}$$

Consider now the complex $B_1 + A_1$. We have, since A_1 is invariant in A, and by means of (12),

* i. e., the order of the subalgebra containing the largest number of independent units.

† This theorem is implicitly contained in MOLIEN's paper, loc. cit., p. 96.

‡ The γ_{ijj} of K are obtained by writing every $\gamma_{ijk} = 0$ in the γ_{ijl} of B.

$$A(B_1 + A_1) = (B + A_1)(B_1 + A_1) \equiv B_1 + A_1;$$

and similarly

$$(B_1 + A_1)A \equiv B + A_1.$$

Thus $B_1 + A_1$ is an invariant subalgebra of A and since $B_1 \frown A_1 = 0$, it follows that $B_1 + A_1 > A_1$ which is impossible since A_1 is maximal by hypothesis. Therefore K is simple.

The chief series of an algebra. The series $E, P_1, P_2, \cdots$ is said to be a chief or principal series (chief composition series) when P_s is a maximal subalgebra of P_{s-1} which is invariant in $E(P_0 = E)$. For the chief series it is shown in exactly the same manner as for the normal series that *the system of complementary algebras* $C_1, C_2, C_3, \cdots$ *is invariant, apart from the sequence* (C_s *being complementary to* P_s *with respect to* P_{s-1}).

We can now define a series of chief indices of composition analogous to the indices of composition above, and show that for a given algebra E, *the system of chief indices of composition is independent of the choice of the chief series, apart from the sequence.*

THE UNIVERSITY OF CHICAGO,
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ON A DEFINITION OF ABSTRACT GROUPS*

BY

ELIAKIM HASTINGS MOORE

In a paper entitled *A definition of abstract groups* (Transactions, vol. 3, pp. 485–492, October, 1902), my second definition (l. c., p. 490)

$$(M'') = (1, 2', 3', 3'', 3''_r, 4''_l)$$

involves postulates not mutually independent. I shall prove here (as stated in October, 1904, Transactions, vol. 5, p. 549) that $(3''_r)$ is redundant, that in the new definition

$$(\overline{M}''): (1, 2, 3'', 3''_l, 4''_l)$$

the postulates are mutually independent, and that this mutual independence remains even for the system

$$(M''_A): (1, 2, 3'', 3''_l, 4''_l, A)$$

defining an abelian group, obtained by adding to those of (M'') the postulate (A) that the multiplication or composition of two elements is commutative.

We have for consideration a set † or class (K) of elements and a multiplication-table or rule of combination $(\circ)$ whereby to every two elements a, b taken in the definite order a, b there corresponds a definite so-called product, in notation $a \circ b$, or, when without confusion, more simply, ab; this product may or may not be an element of the class. The postulates in question are then the following:

(1) If a and b are elements, then ab is an element of the class.

(A) If a and b are elements such that ab and ba are elements, then $ab = ba$.

(2) If a, b, c are elements such that $ab, bc, (ab)c, a(bc)$ are elements, then $(ab)c = a(bc)$.

$(3'')$ There exists an element a such that $aa = a$.

$(3''_l)$ If a and b are elements and $aa = a$, then $ab = b$.

$(3''_r)$ If a and b are elements and $aa = a$, then $ba = b$.

$(4''_l)$ If a and b are elements and $aa = a$, then there exists an element $b_l^{(a)}$ such that $b_l^{(a)} b = a$.

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† In the first definition (M) there was the underlying understanding (l. c., p. 485, footnote ‡) or postulate (0) that the class contain at least one element, and this carried over by implication to (M''), where however in view of $(3'')$ it was not needed; it is now omitted.

If a is an element such that $aa = a$ we designate those parts of $(3''_l)$, $(3''_r)$, $(4''_l)$ which refer to the element a by $(3''^a_l)$, $(3''^a_r)$, $(4''^a_l)$ respectively. Then we prove that $(3''_r)$ is deducible from $(1, 2, 3'', 3''_l, 4''_l)$ in that we prove that $(3''^a_r)$ is deducible from $(1, 2, 3''^a_l, 4''^a_l)$. This fact is proved by a suitable modification of the method used p. 486, 7° loc. cit. If a and b are elements and $aa = a$, then we designate by b', b'' elements, which by $(4''^a_l)$ surely exist, such that $b'b = a$, $b''b' = a$ and have in virtue of $(1, 2, 3''^a_l)$ the continued equality,

$$ba = bb'b = abb'b = b''b'bb'b = b''ab'b = b''b'b = ab = b,$$

that is, $ba = b$, which was to be proved. *

The proof that the postulates of (M''_A) are independent covers the independence of the postulates of (M'') and appears from the following proof systems $(K, \circ)$:

For (1). $K =$ all integers $0, \pm 1, \pm 2, \cdots$ except ± 1. $\circ = +$.

For (2). $K =$ an element a and any class of (at least three) elements x distinct from a. $a \circ a = a$. $a \circ x = x \circ a = x$. $x_1 \circ x_2 = a$ if $x_1 \neq x_2$; $x_1 \circ x_1 = x_3$ where, for every x_1, x_3 is any x except x_1.

For $(3'')$. $K =$ all positive integers. $\circ = +$.

For $(3''_l)$. $K =$ a class of (at least two) elements x. $x_1 \circ x_2 = a$, where a is a definite element, the same for all pairs x_1, x_2.

For $(4''_l)$. $K =$ all positive integers and 0. $\circ = +$.

For (A). $(K, \circ) =$ any non-abelian group.

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* [A second continued equality

$$ba = aba = b''b'ba = b''aa = b''a = b''b'b = ab = b$$

should be noticed.]

NOTE ON THE DEFINITIONS OF ABSTRACT GROUPS AND FIELDS BY SETS OF INDEPENDENT POSTULATES*

BY

EDWARD V. HUNTINGTON

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Introduction.

The theory of groups † was first studied in connection with the algebraic solution of equations, the leaders in the early development being LAGRANGE (1770), RUFFINI (1799), CAUCHY (1814), ABEL (1824), and especially GALOIS (1831). The first discussion of the theory from an abstract point of view was by CAYLEY ‡ in 1854, and the earliest explicit sets of postulates for abstract groups were

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† The most recent bibliography is given by B. S. EASTON, *The constructive development of group-theory*, Philadelphia, 1902; and the most recent text-book is the *Éléments de la théorie des groupes abstraits*, by J. A. DE SÉGUIER, Paris, 1904.

Cf. also G. A. MILLER's two reports on recent progress in group-theory, Bulletin of the American Mathematical Society, vol. 5 (1898-1899), pp. 227–251, and vol. 7 (1900–1901), pp. 121–130.

‡ A. CAYLEY, Philosophical Magazine, vol. 7 (1854), p. 40; see also Proceedings of the London Mathematical Society, vol. 9 (1878), p. 126, or American Journal of Mathematics, vol. 1 (1878), p. 51. [Collected Papers, vol. 2, p. 123; vol. 10, p. 324, or p. 401.] Cf. W. Dyck, Mathematische Annalen, vol. 20 (1882), p. 1.

given by KRONECKER * in 1870 and WEBER † in 1882. WEBER's definition, as finally formulated, ‡ is substantially as follows:

A. There is a rule of combination among the elements of the given set, such that every two elements, a and b, determine uniquely a third element c, called the result of the composition; in symbols: $ab = c$.

B. The associative law holds throughout the set; that is, $(ab)c = a(bc)$.

C. 1) If $ab = ab'$, then $b = b'$.

2) If $ab = a'b$, then $a = a'$.

D. 1) Given a and c, there is an element b such that $ab = c$.

2) Given b and c, there is an element a such that $ab = c$.

E. In the case of abelian groups, the commutative law also holds; that is $ab = ba$.

WEBER notes that if the group is finite, D is a consequence of A, B, and C.

This definition was somewhat simplified by BURNSIDE § in 1897, and, more explicitly, by Professor PIERPONT ‖ in 1900. PIERPONT replaced conditions C and D by the following:

C'. There exists a unique element 1, called the identity, such that $a1 = 1a = a$ for every element a.

D'. For every element a there exists an element a^{-1}, called the inverse of a, such that $aa^{-1} = a^{-1}a = 1$.

The equivalence of the two definitions is readily established. PIERPONT's definition, like most of the earlier definitions, was stated only for the case of finite groups; but essentially the same definition had been used for groups in general by Professor MOORE in lectures ¶ in 1897.

The earliest discussion of the *independence* of the postulates for abstract groups was, as far as I know, that contained in a paper of my own ** in 1902. I noticed in the first place that WEBER's postulate C was deducible from his A, B, and D, †† and further, that by a peculiar wording of the associative law,

* L. KRONECKER, Monatsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin (1870), p. 882.

† H. WEBER, Mathematische Annalen, vol. 20 (1882), p. 302. Cf. also G. FROBENIUS in Crelle's Journal für die reine und angewandte Mathematik, vol. 100 (1887), p. 179, and in the Sitzungsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin (1895), p. 163.

‡ H. WEBER, Mathematische Annalen, vol. 43 (1893), p. 521; or *Algebra*, second edition, vol. 2 (1899), p. 3.

§ W. BURNSIDE, *Theory of groups of finite order*, 1897, p. 11.

‖ J. PIERPONT, Annals of Mathematics, ser. 2, vol. 2 (1900–01), p. 47. (From a course of lectures delivered in 1896 at the Buffalo Colloquium of the American Mathematical Society.)

¶ Cf. E. H. MOORE, Transactions, vol. 3 (1902), p. 488, footnote.

** E. V. HUNTINGTON, Bulletin of the American Mathematical Society, vol. 8 (1901–02), pp. 296–300, revised in Transactions, vol. 4 (1903), p. 30. See also a second definition, in Bulletin, loc. cit., pp. 388–391, and two corresponding definitions for abelian groups, in Transactions, loc. cit., pp. 27–29.

†† For the case of Abelian groups, cf. § 4, below.

postulate A also could be made redundant. Since postulate A is the fundamental postulate of the whole theory, the resulting definition (comprising only D and a modified form of B) had obvious disadvantages. Revised forms of this definition, however, in which the rule of combination is replaced by a relation, as suggested by Professor BÔCHER* in his address at the St. Louis Congress of 1904, have considerable logical interest, and will be presented in §§ 5–6 of this note.

More convenient definitions by sets of independent postulates were given in the same year (1902) by Professor MOORE.† The first of these definitions is of the type published by PIERPONT, differing from WEBER's definition in the use which it makes of the identity and the inverse elements. A second definition given at the end of the paper is a modification of the first, embodying a further analysis of the identical element, and demanding, in particular, the existence of an "idempotent" element i such that $ii = i$. A note by MOORE on this second definition appears in the present number of the Transactions.

Another modification of MOORE's first definition, reverting more nearly to PIERPONT's original form, has been recently given by Professor DICKSON, and will also be found in this number of the Transactions.

In §1 of the present note I propose a further modification of the PIERPONT-MOORE type of definition, in the direction indicated by the latter part of MOORE's paper. Although the points of difference are so slight as to seem almost trivial, yet the introduction of postulate 4, demanding explicitly the uniqueness of the identical element, and the "weakened" forms in which postulates 5 and 6 are now stated, will be found very convenient when one has to test a given system for the group property.‡

Closely connected with the theory of groups is the theory of fields, § suggested by GALOIS, and due, in concrete form, to DEDEKIND ‖ in 1871. The word *field* is the English equivalent for DEDEKIND's term *Körper*: KRONECKER's term *Rationalitätsbereich*, ¶ which is often used as a synonym, had originally a some-

* M. BÔCHER, Bulletin of the American Mathematical Society, vol. 11 (1904–05), p. 126, footnote.

† E. H. MOORE, Transactions, vol. 3 (1902), pp. 485–492, revised in the present number of the Transactions, p. 179.

‡ Further analysis of the postulates, such as I have attempted in Transactions, vol. 6 (1905), pp. 34–36, does not seem likely to lead to practical advantage, except possibly in the case of the associative law.

§ See parts of *Linear Groups, with an exposition of the Galois Field theory*, by L. E. DICKSON, in the TEUBNER series of mathematical text-books, 1901; and parts of the *Éléments de la théorie des groups abstraits*, by J. A. DE SÉQUIER, Paris, 1904. [See review by DICKSON in the Bulletin of the American Mathematical Society, vol. 9 (1904–05), pp. 159–162.]

‖ P. G. L. DIRICHLET, *Vorlesungen über Zahlentheorie*, edited, with supplementary material, by R. DEDEKIND; 2d edition (1871), p. 424; 4th edition (1894), p. 452.

¶ L. KRONECKER, *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, 1882, Crelle, vol. 92 (1882), p. 1; Werke, vol. 2, p. 237. Cf. J. KÖNIG, *Einleitung in die allgemeine Theorie der algebraischen Gröszen*, 1903, p. 183.

what different meaning. The earliest expositions of the theory from the general or abstract point of view were given independently by WEBER* and by MOORE,† in 1893, WEBER's definition of an abstract field being substantially as follows:

I. There are two rules of combination among the elements of the given set, the one called addition, and the other multiplication.

II. The set forms a group with respect to addition (see conditions A, B, C, D, above). The result of the composition of two elements is here called their sum $(a+b)$, the identity of the group being called zero (0), and the inverse operation subtraction.

III. If the identity of the additive group is excluded, the remaining elements form a group with respect to multiplication. The result of the composition of two elements is here called their product ($a \times b$, or $a \cdot b$, or ab), the identity being called unity (1), and the inverse operation division (the divisor being different from zero).

IV. Addition and multiplication are both commutative.

V. Addition is distributive with respect to multiplication; that is, $a(b+c)=(ab)+(ac)$.

VI. $a(-b)=-(ab)$.

As consequences of this definition,‡ WEBER proves that $a \times 0 = 0 \times a = 0$ for every a, and that a product cannot be zero unless at least one of its factors is zero; but he makes no attempt to free the definition from redundant statements.

The earliest sets of *independent* postulates for abstract fields were given in 1903 by Professor DICKSON§ and myself;‖ all these sets were natural extensions of the sets of independent postulates that had already been given for groups.

In § 2 of the present note I propose another set of postulates for fields, based on the postulates for groups given in § 1, and possessing similar advantages.

Still other sets will be found in § 4, and in the paper by Professor DICKSON (already referred to) in the present number of the Transactions.

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* H. WEBER, Mathematische Annalen, vol. 43 (1893), p. 526.

† E. H. MOORE, Mathematical Papers read at the International Mathematical Congress at Chicago in 1893 (published in 1896), p. 210; abstract in the Bulletin of the New York Mathematical Society, vol. 3 (1893–94), p. 75. MOORE's theorem that every existent finite field is the abstract form of some Galois Field is established in this paper; but the term *Galois Field* as here used must not be confused with the term *Galois'scher Körper*, introduced by DEDEKIND (loc. cit., 2d edition, p. 455).

‡ The most familiar and important example of an infinite field is furnished by the rational numbers, under the operations of ordinary addition and multiplication. In fact, a field may be briefly described as a system in which the rational operations of algebra may all be performed (excluding division by zero). A field may be finite, provided the number of elements (called the order of the field) is a prime or a power of a prime.

§ L. E. DICKSON, Transactions, vol. 4 (1903), pp. 13–20.

‖ E. V. HUNTINGTON, Transactions vol. 4 (1903), pp. 31–37.

§ 1. *A definition of groups and abelian groups.*

As already stated, the definition of abstract groups contained in this section is a modification of the definition given by Professor MOORE in 1902.

We consider a class K, and a rule of combination, subject to the following conditions (the result of combining a and b being denoted by ab):

POSTULATE 1. If a and b are elements of the class, then there is an element c in the class such that $ab = c$; and this element c is uniquely determined by a and b.*

POSTULATE 2. The associative law holds throughout K; that is,

$$(ab)c = a(bc),$$

whenever a, b, c, ab, bc, $(ab)c$, and $a(bc)$ are elements of K.

POSTULATE 3. There is at least one element i such that $ii = i$.

POSTULATE 4. There is not more than one element i such that $ii = i$; that is, if x and y are elements such that $xx = x$ and $yy = y$, then $x = y$.

POSTULATE 5. If there is a unique element i such that $ii = i$, then either $ia = a$ for every element a, or else $ai = a$ for every element a.

POSTULATE 6. If there is a unique element i such that $ii = i$, then for every element a there is either an element a_r such that $aa_r = i$, or else an element a_l such that $a_l a = i$.

From these postulates 1–6 the following theorems can be deduced, the proofs for which will be given in § 7:

Theorem I. If i is the unique element described in postulates 3 and 4, then $ia = ai = a$ for every element a.

Theorem II. If $ab = ab'$, then $b = b'$; if $ab = a'b$, then $a = a'$.

From these theorems we have at once two further theorems, by the aid of postulate 6:

Theorem III. Every element a determines uniquely an element a^{-1} such that $aa^{-1} = a^{-1}a = i$, where i is the element described in postulates 3 and 4.

Theorem IV. Every two elements a and b determine uniquely an element x,

* If c were not uniquely determined by a and b, then the expression ab would have to be regarded as a multiple-valued function of a and b, and the assertion: $ab = c$, would mean merely that *one of the values* of ab is equal to c. Similarly, $(ab)c$ would mean the multiple-valued function obtained by combining (ab), itself multiple valued, with the element c; and the equation $(ab)c = a(bc)$ would assert merely that *one* of the values of $(ab)c$ is equal to *one* of the value of $a(bc)$.

With this understanding of the notation, a system which satisfies all the postulates except the last half of (1) may be constructed as follows: Let $K =$ any finite or infinite class which contains a special element X and at least one other element; and define the rule of combination so that if $a \neq b$, ab may have any value within the class; while if $a = b$, $aa = X$.

Hence the second part of postulate 1 is essential to the definition. In fact, it is only the uniqueness of the symbol ab which gives this notation any advantage over the relational notation employed in §§ 5–6.

namely $x = a^{-1}b$, such that $ax = b$; and also an element y, namely $y = ba^{-1}$, such that $ya = b$.

Thus we see that any system which satisfies the postulates 1–6 will be a *group* with respect to the given rule of combination. The element i is the *identity* of the group, and the element a^{-1} is the *inverse of* a.

If we wish to make the group *abelian*, we must add another postulate, namely:

POSTULATE 7 (for abelian groups). The commutative law holds throughout K; that is,

$$ab = ba,$$

whenever a, b, ab, and ba are elements of K.

These seven postulates are independent, as will be shown by the examples constructed in § 3; so that no one of them can be deduced from the remaining six.

If we wish to make the group *finite*, we must add the following postulate:

POSTULATE N (for finite groups). The number of elements in the class is some positive integer, n.

After the introduction of this postulate N, postulates 3 and 6 become redundant (see § 7), so that *the five postulates*

$$1,\ 2,\ 4,\ 5,\ \textit{and}\ N$$

are sufficient for finite groups. The independence of these postulates, when $n > 2$, is established in § 3.

§ 2. *A definition of fields.*

The following set of postulates for abstract fields is suggested immediately by the postulates given in § 1 for groups.

We consider a class K, and two rules of combination, called addition and multiplication, subject to the following conditions ($A1 - A6$, $M1 - M4$, $M6$, $M7$, and D):*

POSTULATE $A1$. If a and b are elements of the class, then their "sum" $a + b$, is an element of the class, and is uniquely determined by a and b.

POSTULATE $A2$. The associative law for addition holds throughout the class:

$$(a + b) + c = a + (b + c).$$

POSTULATE $A3$. There is at least one element z such that $z + z = z$.

POSTULATE $A4$. There is not more than one element z such that $z + z = z$.

POSTULATE $A5$. If there is a unique element z such that $z + z = z$, then either $z + a = a$ for every element a, or else $a + z = a$ for every element a.

* The letters A and M in the designation of the postulates are intended to suggest addition and multiplication respectively, while D indicates the distributive law for addition with respect to multiplication.

POSTULATE $A6$. If there is a unique element z such that $z + z = z$, then for every element a there is either an element a_r' such that $a + a_r' = z$, or else an element a_l' such that $a_l' + a = z$.

From these postulates $A1$–$A6$ we have at once, by § 1, the following theorem:

Theorem 1. *The class K is a group with respect to addition.*

The identity, z, of this group, is called the *zero-element* of the field, and is denoted by 0; while the inverse of an element a is here called the *negative of* a, and is denoted by $-a$.

POSTULATE $M1$. If a and b are elements of the class, then their "product," $a \times b$ (or $a \cdot b$, or ab), is an element of the class, and is uniquely determined by a and b.

POSTULATE $M2$. The associative law for multiplication holds throughout the class:

$$(a \times b) \times c = a \times (b \times c).$$

POSTULATE $M3$. There is at least one element u such that $u \times u = u$ and $u + u \neq u$.

POSTULATE $M4$. There is not more than one element u such that $u \times u = u$ and $u + u \neq u$.

Lemma $M5$. If there is a unique element u such that $u \times u = u$ and $u + u \neq u$, then either $u \times a = a$ for every element a, or else $a \times u = a$ for every element a.

This lemma $M5$ is not included in the list of postulates, since it will prove to be deducible from $A1$–$A6$, $M1$–$M4$, $M6$, $M7$, and D (see § 7); it is assumed for the moment, however, as are also lemmas $M7'$ and D', below, in order to show exactly how much of the postulates $M7$ and D is required for the proof of theorem 3.

POSTULATE $M6$. If there is a unique element u such that $u \times u = u$ and $u + u \neq u$, then for every element a, provided $a + a \neq a$, there is either an element a_r'' such that $a \times a_r'' = u$, or else an element a_l'' such that $a_l'' \times a = u$.

Lemma $M7'$. If 0 is the zero-element of the system (theorem 1), and u the element in postulates $M3$–$M4$, then $0 \times u = u \times 0$.

This lemma will appear as a special case of postulate $M7$.

Lemma D'. If 0 is the zero-element of the system (theorem 1), then either $0 \times a = 0$ for every element a, or else $a \times 0 = 0$ for every element a.

This lemma will prove to be an immediate consequence of postulate D, in view of $A3$–$A4$, taking $b = c = 0$.

The propositions $A1$–$A6$, $M1$–$M6$, $M7'$, and D' are now sufficient (see § 7) to establish theorems 2–3, without requiring the more general postulates $M7$ and D; and until these postulates $M7$ and D are introduced, the fourteen propositions just mentioned are all independent (see § 3).

The theorems in question concern the group-property of the field with regard to multiplication, as follows:

Theorem 2. $a \times 0 = 0 \times a = 0$, *for every element* a.

Theorem 3. *If the zero-element is excluded, the remaining elements form a group with respect to multiplication.*

The identity, u, of this group is called the *unit-element* of the field and is denoted by 1; while the inverse of any element a in this group is called the *reciprocal of* a and is denoted by $1/a$.

The remaining postulates for a field are the following:

POSTULATE $M7$. The commutative law for multiplication holds throughout the class; that is,

$$a \times b = b \times a.$$

POSTULATE D. Either the left-hand or else the right-hand distributive law for multiplication with respect to addition holds throughout the class; that is, either

$$a \times (b + c) = (a \times b) + (a \times c),$$

or else

$$(b + c) \times a = (b \times a) + (c \times a).$$

From $M7$ and D, in view of $A1$ and $M1$, we have at once:

Theorem 4. *Multiplication is commutative and distributive.*

To show that $a \times (-b) = -(a \times b)$, we have only to put $b + c = 0$ in postulate D, using $M7$ and theorem 2.

Finally, by a method due to HILBERT (see § 7), we can deduce the commutative law for addition:

Theorem $A7$. *Throughout the class,* $a + b = b + a$.

Thus we see that any system $(K, +, \times)$ which satisfies the postulates $A1$–$A6$, $M1$–$M4$, $M6$, $M7$, and D, will be a field with respect to the rules of combination $+$ and $\times$.

All these postulates are independent, as will be shown by the examples constructed in § 3.

§ 3. *Proof of the independence of the postulates of* §§ 1–2.

In this section, the symbols $+$, $\times$, 0, and 1 are used only in their ordinary arithmetical meanings, the general operations of "addition" and "multiplication" being denoted by $\oplus$ and $\odot$, and the "zero-" and "unit-" elements by z and u respectively. *

The independence of the thirteen postulates of § 2, for abstract fields, is

* The symbols $\oplus$ and $\odot$ were first used in this connection in Transactions, vol. 4 (1903), p. 31; cf. vol. 5 (1904), p. 292, and vol. 6 (1905), pp. 19 and 22.

established by the existence of the following systems $(K, \oplus, \odot)$, each of which satisfies all the other postulates, but not the one for which it is numbered.

The first six of these systems, together with any non-abelian group, serve also to show the independence of the seven postulates of § 1 for abstract groups.

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*For A*1. $K =$ all real numbers; $a \oplus b = a + b$ when a or b or $a + b$ is zero, otherwise $a \oplus b$ not in the class; $a \odot b = a \times b$.

*For A*2. $K =$ all real numbers; $a \oplus b = a + b$ except that $a \oplus a = 0$; $a \odot b = a \times b$.

*For A*3. $K =$ all positive real numbers; $a \oplus b = a + b$; $a \odot b = a \times b$.

*For A*4. $K =$ all real numbers, together with an extra element ∞; $a \oplus b = a + b$ and $a \odot b = a \times b$, understanding that $\infty \oplus \infty = \infty$, $\infty \oplus a = a \oplus \infty = \infty$, and $a \odot \infty = \infty$, $\infty \odot a = a \odot \infty = \infty$. Here $z = 0$ or ∞ and $u = 1$.

*For A*5. $K =$ all real numbers; $a \oplus b = 0$; $a \odot b = a \times b$.

*For A*6. $K =$ all positive real numbers with 0; $a \oplus b = a + b$; $a \odot b = a \times b$.

*For M*1. $K =$ all real numbers; $a \oplus b = a + b$; $a \odot b = a \times b$ when a or b is 1 or when $a \times b$ is 1 or 0, otherwise $a \odot b$ not in the class.

*For M*2.* $K =$ all complex numbers of the form $(a, \beta, \gamma) = a + \beta i + \gamma j$, where a, β, and γ, are real numbers; $\oplus = +$; $\odot = \times$, the multiplication-table for the three principal units being the following:

$\odot$	1	i	j
1	1	i	j
i	i	-1	1
j	j	1	-2

Here $z = (0, 0, 0)$ and $u = (1, 0, 0)$. To find the reciprocal, (X, Y, Z) of (a, β, γ), take $X = a/\Delta$, $Y = -\beta/\Delta$, $Z = -\gamma/\Delta$, where $\Delta = a^2 + (\beta - \gamma)^2 + \gamma^2$. The associative law fails, since $(ii)j \neq i(ij)$.

*For M*3. $K =$ all even integers (positive, negative, and zero); $\oplus = +$; $\odot = \times$.

*For M*4. $K =$ all complex numbers (a, β), where a and β are real; $\oplus = +$; $(a_1, \beta_1) \odot (a_2, \beta_2) = (a_1 a_2, \beta_1 a_2 + a_1 \beta_2 + \beta_1 \beta_2)$.

Here $z = (0, 0)$, and $u = (1, 0)$ or $(0, 1)$ or $(1, -1)$.

*For M*6. $K =$ all integers; $\oplus = +$; $\odot = \times$. The following system may also be used: $K =$ all complex numbers (a, β), where a and β are real; $\oplus = +$; $(a_1, \beta_1) \odot (a_2, \beta_2) = (a_1 a_2, \beta_1 a_2 + a_1 \beta_2)$.

* This system was suggested to me by Professor DICKSON.

For M7. $K =$ all quaternions, $a + \beta i + \gamma j + \delta k$, where a, β, γ, and δ are real numbers; $\oplus = +$; $\odot = \times$.

For D. $K =$ all integers; $a \oplus b = a + b$; $0 \odot a = a \odot 0 = 0$; when $a \neq 0$ and $b \neq 0$, $a \odot b$ is defined as follows: let $A = a$ or $a + 1$, according as a is positive or negative, and let $B = b$ or $b + 1$, according as b is positive or negative; then $a \odot b = A + B$ when $A + B$ is positive, and $a \odot b = A + B - 1$ when $A + B$ is negative or zero.

The independence of the postulates is thus established. Since each of the systems is infinite, the postulates will remain independent even when we add the demand that the group or field shall be infinite.

In regard to the lemmas $M5$, and $M7'$, and D', we notice that all the systems just given satisfy them; further, the following systems show that they cannot be proved without the aid of $M7$ or D.

For M5 and D. Let $K =$ all real numbers; $\oplus = +$; $a \odot b = 1$, except that $a \odot 0 = 0 \odot a = 0$.

For M7' and M7. $K =$ a class of two elements, 0 and 1, with $a \oplus 0 = 0 \oplus a = a$, $1 \oplus 1 = 0$, and $a \odot b = a$; or the same system, with $a \odot b = b$. No system of this kind exists that has more than two elements.

For D' and D. Let $K =$ all real numbers; $\oplus = +$; $a \odot b = a + b - 1$.

It remains to prove the independence of the postulates 1, 2, 4, 5, N, for finite groups* of order $n > 2$ (§1).† To show the independence for the first four cases, let $K =$ the class of n integers from 0 to $n - 1$ inclusive, and define the rule of combination as follows:

For 1. $a \oplus b = 0$ when $a + b = n$, and $a \oplus 0 = 0 \oplus a = a$; otherwise $a \oplus b$ not in the class.

For 2. $a \oplus b = 0$ except that $a \oplus 0 = 0 \oplus a = a$.

For 4. $a \oplus b = 0$ except that $a \oplus a = a$.

For 5. $a \oplus b = 0$.

To show the independence of postulate N, consider any infinite group, or an empty class K.

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* I have not attempted to get a set of independent postulates for *finite fields*. All that has been done in this direction will be found in Professor DICKSON'S paper in the present number of the Transactions.

† If $n = 1$, postulate 1 is sufficient; if $n = 2$, postulates 1, 2, 4, 5 are independent, the non-group systems for 1 and 2 being the following:

	0	1
0	1	X
1	X	0

and

	0	1
0	1	1
1	1	0

In regard to the independence of the commutative law in the case of finite groups of order n, Professor DICKSON has just proved the following interesting theorem: Let $n = A \cdot B \cdot C \cdots$, where $A = a^\alpha$, $B = b^\beta$, $C = c^\gamma$, $\cdots$, the $a, b, c, \cdots$, being distinct primes; then

(1) if any one of the exponents $\alpha, \beta, \gamma, \cdots$ is greater than 2, there is a non-abelian group of order n; and

(2) if any one of the expressions $A-1, B-1, C-1, \cdots$ is divisible by any one of the primes $a, b, c, \cdots$, there is a non-abelian group of order n.

If neither of these conditions is satisfied, every group of order n is abelian.

§ 4. *A shorter definition of a field.*

This section contains a shorter definition of a field, by means of ten postulates, the independence of which, however, is not completely established.*

POSTULATES 1–7. The same as $A1, A2, A7; M1, M2, M7; D$. (See § 2.)

POSTULATE 8. Given a and b, there is an x such that $a + x = b$.

POSTULATE 9. Given a and b, and $a+a \neq a$, there is a y such that $a \times y = b$.

POSTULATE 10. There are at least two elements in the class.

Five of these postulates, namely 1, 2, 3, 8, and 10, constitute a definition of abelian groups; but for the purpose of testing a given system, neither of these shorter definitions is so convenient as those given in § 1.

The following theorems, proved in § 7, show that any system which satisfies the postulates 1–10 is a field with respect to $+$ and $\times$:

Theorem I. There is a unique element 0 such that $0 + a = a + 0 = a$ for every element a. (From 1, 2, 3, 8, 10.)

Theorem II. If $a + b = a + b'$, then $b = b'$. (From 1, 2, 3, 8, 10.)

Theorem III. There is a unique element 1, different from 0, such that $1a = a1 = a$ for every element a.

Theorem IV. If $a \times b = a \times b'$, and $a \neq 0$, then $b = b'$.

Theorem V. If $a \neq 0$ and $b \neq 0$, then $a \times b \neq 0$.

The independence of all the postulates, except 1 and 4, is established by the following systems:

[*For* 1 *and* 8. K = all real numbers; $a \oplus b = a + b$ when a and b are integers, otherwise $a \oplus b$ not in the class; $\odot = \times$.]

For 2. K = all real numbers; $a \oplus b = 2(a + b)$; $\odot = \times$.

For 3. K = all real numbers; $a \oplus b = b$; $\odot = \times$.

[*For* 4 *and* 9. K = all real numbers; $\oplus = +$; $a \odot b = a \times b$ when a and b are rational, otherwise $a \odot b$ not in the class.]

For 5. K = all complex numbers (α, β), where α and β are real; $\oplus = +$; $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, -\alpha_1\beta_2 - \alpha_2\beta_1)$.

* Compare a similar set of thirteen postulates, all independent, in a paper of mine on complex algebra, in the present number of the Transactions.

In the geometrical representation of this system, $\odot$ is the ordinary multiplication followed by reflection in the axis of reals.

For 6. K = all real numbers; $\oplus = +$; $a \odot b = b$.

For 7. K = all real numbers; $\oplus = +$; $\odot = +$.

For 8. K = all positive real numbers; $\oplus = +$; $\odot = \times$.

For 9. K = all integers; $\oplus = +$; $\odot = \times$.

For 10. K = an empty class; or K = a class containing a single element a, with $a \oplus a = a$ and $a \odot a = a$.

The independence of postulates 1 and 4 is still an open question, unless postulates 8 and 9 are artificially "weakened".

§§ 5–6. *Definitions of groups in terms of a triadic relation, R.*

The sets of postulates for abstract groups given in §§ 5–6 are modifications of the writer's first definition of 1902 (loc. cit.), expressed in terms of the triadic relation suggested by Professor BÔCHER (loc. cit.).

The fundamental concepts are here a *class*, K, and a *relation*, R; instead of writing "$ab = c$," which means "the combination of a and b equals c," we write "$R(abc)$," which means "the three elements a, b, and c satisfy the given relation R."

The use of this notation suggested the more explicit division of the first postulate of § 1 into its two component statements, namely postulates IV in § 5, and postulate IV′ in § 6.

§ 5.

One set of postulates for which, however, the proofs of independence are not complete, is the following (a, b, c, etc. denoting elements of K):

POSTULATE I. The class K is not an empty class.

POSTULATE II. Given b and c there is an a such that $R(abc)$.

POSTULATE III. Given a and c there is a b such that $R(abc)$.

POSTULATE IV. Given a and b there is a c such that $R(abc)$.

POSTULATE V. If $R(abp)$, $R(pcM)$, $R(bcq)$, and $R(aqN)$, then $M = N$.

From these postulates I–V the following theorems are deduced (see § 7):

Theorem 1. If $R(abc)$ and $R(a'bc)$, then $a = a'$.

Theorem 2. If $R(abc)$ and $R(ab'c)$, then $b = b'$.

Theorem 3. If $R(abc)$ and $R(abc')$, then $c = c'$.

If now we write $ab = c$ in place of $R(abc)$, postulate V gives us the associative law: $(ab)c = (p)c = M$; $a(bc) = a(q) = N$; and all the conditions in WEBER's definition of a group are clearly satisfied.

The independence of all the postulates except IV is established by the systems used in § 6; but the question of the independence or deducibility of postulate IV is undecided.

§ 6.

Another set of postulates, less symmetrical than that given in § 5, but admitting complete proofs of independence, is the following:

POSTULATES I–III. The same as in § 5.

POSTULATE IV′. If $R(abc)$ and $R(abc')$, then $c = c'$.

POSTULATE V′. Either (1): If $R(abp)$, $R(bcq)$, and $R(pcM)$, then $R(aqM)$; or else (2): If $R(abp)$, $R(bcq)$, and $R(aqN)$, then $R(pcN)$.

From these postulates we can prove (see § 7) the theorems 1–2 of § 5, and also

Theorem 3′. Given a and b there is a c such that $R(abc)$.

Hence the two definitions are clearly equivalent.

The proofs of independence are as follows:

For I. K = an empty class.

For II–III. K = any class of more than two elements; for II, let $R(abc)$ mean $b = c$; for III, let $R(abc)$ mean $a = c$.

For IV′. K = any class of more than two elements, with $R(abc)$ true for all values of a, b, and c.

For V′. K = all real (or all rational) numbers, with $R(abc)$ signifying $a + b = 2c$; or again, K = the n positive integers ($n > 2$) from 0 to $n - 1$ inclusive, with $R(abc)$ holding whenever $c = n - s$, where s is congruent to $a + b$ modulo n. In this last system, $(11)2 \neq 1(12)$.

It may be noticed that the postulates I–III, IV, and V′ are independent,* but not sufficient to define a group, as witness the systems in which $R(abc)$ is always true.

The postulates I–III, IV′, and V are likewise independent, but their sufficiency is still an open question.

§ 7. *Proofs of theorems in the preceding sections.*

To avoid interruption in reading, a number of theorems in the preceding sections have been stated without proof; the requisite demonstrations are here supplied.

In § 1: *Proof of theorem* I. If i is the element described in postulates 3 and 4, we have the following lemmas:

Lemma 1. If $ab = i$, then $ba = i$. (By 1–5.) For, if $ab = i$, then $(ba)(ba) = b(ab)a = bia = ba$; whence $ba = i$, by 4.

Lemma 2. For every element a there is at least one element α such that $a\alpha = \alpha a = i$. (By postulate 6 and lemma 1.)

The proof of the main theorem is then as follows: Suppose $ia = a$ for every element a, and take α so that $a\alpha = \alpha a = i$; then

$$ai = i(ai) = i[a(\alpha a)] = i[(a\alpha)a] = i(ia) = (ii)a = ia = a.$$

*To show the independence of IV in this set, let $R(abc)$ signify $a = b$.

Similarly, if we suppose $ai = a$ for every element a, then

$$ia = (ia)i = [(a\alpha)a]\,i = [a(\alpha a)]\,i = (ai)i = a(ii) = ai = a.$$

In § 1: *Proof of theorem* II. By lemma 2, take α so that $\alpha a = i$, and β so that $b\beta = i$. Then if $ab = ab'$,

$$b = ib = (\alpha a)b = \alpha(ab) = \alpha(ab') = (\alpha a)b' = ib' = b';$$

and similarly, if $ab = a'b$,

$$a = ai = a(b\beta) = (ab)\beta = (a'b)\beta = a'(b\beta) = a'i = a'.$$

In § 1: *Proof of postulates* 3 *and* 6 *for finite groups.* (The proof of 3 depends merely on 1, 2, and N; that for 6 requires 1, 2, 4, 5, and N.)

Let a be any element, and form the sequence of elements

$$a_1 = aa,\ a_2 = a_1a_1,\ a_3 = a_2a_2,\ \cdots,\ a_{k+1} = a_ka_k.$$

Since the class is finite, this sequence must eventually contain repetitions; that is, there must be indices p and $p+q$ such that $a_p = a_{p+q}$. The element

$$x = a_pa_{p+1}a_{p+2}\cdots a_{p+q-1}$$

will then be such that $xx = x$, which establishes postulate 3. *

Now this element x is simply the combination of a certain number, say m, † of the a's:

$$aaa\cdots a = i;$$

hence the combination of $m-1$ of the a's will be an element a' such that $aa' = i$, which establishes postulate 6.

In § 2: *Proof* ‡ *of lemma M*5.

a) $a \times 0 = 0$. (From *A*1, *A*3–*A*4, *M*1, *M*7, *D*.) For,

$$a \times 0 = a \times (0 + 0) = (a \times 0) + (a \times 0),$$

whence $a \times 0 = 0$, by *A*4.

b) If $u \times x = 0$, then $x = 0$. (From *A*1, *A*2–*A*4, *M*1–*M*4, *M*6–*M*7, *D*.) For suppose $x \neq 0$, and take x'' by *M*6 so that $x'' \times x = u$. Then by *a*), $u = u \times u = u \times (x \times x'') = (u \times x) \times x'' = 0 \times x'' = 0$, which is impossible.

c) $a \times (b - c) = (a \times b) - (a \times c)$. (From *A*1–*A*6, *M*1–*M*4, *M*6–*M*7, *D*.) For, $a \times (-c) + a \times c = a \times (-c + c) = a \times 0 = 0$, whence

$$a \times (-c) = -(a \times c).$$

* This proof for 3, which suggested the proof for 6, is taken from the paper by E. H. MOORE, loc. cit. (1902), p. 490.

† Here $m = 2^p(2^q - 1)$.

‡ I am indebted to Mr. G. D. BIRKHOFF for this demonstration.

d) If $u \times a = x$, then $u \times x = x$. (From $M1$–$M4$.) For,

$$u \times x = u \times (u \times a) = (u \times u) \times a = u \times a = x.$$

From *c*) and *d*) we can now prove the lemma, as follows: Let $u \times a = x$; then $u \times (a - x) = (u \times a) - (u \times x) = x - x = 0$, whence $a - x = 0$, or $x = a$.

In § 2: *Proof of theorem* 2. (The proof depends on $A3$–$A4$, $M1$–$M5$, $M7'$ and D', but not on $M6$.)

If the first part of D' is true, $0 \times a = 0$ for every element a. Then $(a \times 0) \times (a \times 0) = a \times (0 \times a) \times 0 = a \times 0 \times 0 = a \times 0$, whence, by $M4$, $a \times 0 =$ either 0 or u.

But the second alternative, $a \times 0 = u$, must be rejected, in view of $M7'$. For, suppose $a \times 0 = u$ for any particular value of a. Then $u \times c = (a \times 0) \times c = a \times (0 \times c) = a \times 0 = u$, for every element c, so that first half of $M5$ cannot be true. Therefore, by the second half of $M5$, $c \times u = c$ for every element c. Then, for every c, $c \times c = (c \times u) \times c = c \times (u \times c) = c \times u = c$, whence, by $M4$, every c must be either 0 or u. But if 0 and u are the only elements in the class, then $u \times 0 = 0 \times u$, by $M7'$, or $u \times 0 = 0$.

Therefore $a \times 0 = 0$ for all values of a.

If the second part of D' is true, instead of the first part, the theorem is proved in a similar way.

In § 2: *Proof of theorem* 3.

(1°) If $a \times b = u$, then $b \times a = u$. For, if $a \times b = u$, then $(b \times a) \times (b \times a) = b \times (a \times b) \times a = b \times u \times a = b \times a$, whence, by $M4$, $b \times a$ is either u or 0. But it cannot be 0, since if it were we should have $u \times u = (a \times b) \times (a \times b) = a \times (b \times a) \times b = a \times 0 \times b = 0$ (by theorem 2), which is impossible by $M5$.

(2°) For every element a, provided $a \neq 0$, there is at least one element α such that $a \times \alpha = \alpha \times a = u$ [by (1°) and $M6$]; and any such element α will be $\neq 0$ when $a \neq 0$ [by theorem 2, since $0 \neq u$].

(3°) If $a \neq 0$, then $u \times a = a \times u = a$. [This follows from (1°) and (2°), as in the proof of theorem I in § 1.]

(4°) If $a \neq 0$ and $b \neq 0$, then $a \times b \neq 0$. For, suppose $a \times b = 0$ and $a \neq 0$; then taking α so that $\alpha \times a = u$ we have

$$b = u \times b = (\alpha \times a) \times b = \alpha \times (a \times b) = \alpha \times 0 = 0.$$

Thus all the conditions for a group (see § 1) are satisfied.

In § 2: *Proof* of theorem* $A7$. (The proof depends on $A1$–$A6$, $M1$–$M6$,

*D. HILBERT, *Über den Zahlbegriff*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899–1900), p. 183.

and both parts of D). Since the system is a group with respect to addition, we know that from $a + b = a + b'$ follows $b = b'$, and from $a + b = a' + b$ follows $a = a'$ (theorem 2, § 1). By D, in view of $M7$, we have

$$(a + b) \times (1 + 1) = a \times (1 + 1) + b \times (1 + 1) = a + a + b + b,$$

$$= (a + b) \times 1 + (a + b) \times 1 = a + b + a + b.$$

Therefore $a + a + b = a + b + a$, and hence $a + b = b + a$.

In § 4: *Proof of theorems* I–V.

I. Let c be any fixed element and a any other element. Take z so that $c + z = c$, and x so that $x + c = a$. Then $a+z=(x+c)+z=x+(c+z)=a$. Further, if $a + z_1 = a$ for every a, and also $a + z_2 = a$ for every a, then $z_2 + z_1 = z_2$ and $z_1 + z_2 = z_1$, whence by postulate 3, $z_1 = z_2 = 0$.

II. Take x so that $x+a=0$; then $b=0+b=x+a+b=x+a+b'=0+b'=b'$.

III. Let c be any fixed element not 0, and a any element not 0. Take u so that $cu = c$ and y so that $yc = a$. Then $au = (yc)u = y(cu) = yc = a$.

IV. Take y so that $ya = u$; then $b = ub = yab = yab' = yb' = b'$.

V. Suppose $ab = 0$ and $a \neq 0$, and take y so that $ya = 1$. Then $b = 1b = yab = y0 = 0$.

In § 5: *Proof of theorem* 1. Given $R(abc)$ and $R(a'bc)$.

Take z and x so that $R(za'a)$ and $R(zcx)$; and y so that $R(cya')$. Then from $R(za'a)$, $R(abc)$, $R(a'bc)$, $R(zcx)$, follows $c = x$; and from $R(zcc)$, $R(cya')$, $R(cya')$, $R(za'a)$, follows $a' = a$.

In § 5: *Proof of theorem* 2. Given $R(abc)$ and $R(ab'c)$.

Take z and x so that $R(b'zb)$ and $R(czx)$: and y so that $R(ycb')$. Then from $R(ab'c)$, $R(czx)$, $R(b'zb)$, $R(abc)$, follows $c = x$; and from $R(ycb')$, $R(b'zb)$, $R(czc)$, $R(ycb')$, follows $b = b'$.

In § 5: *Proofs of theorem* 3. Given $R(abc)$ and $R(abc')$.

First proof (using only II, IV, and V): Take x, β, and y so that $R(xab)$, $R(ax\beta)$, and $R(\beta ay)$. Then from $R(ax\beta)$, $R(\beta ay)$, $R(xab)$, $\left\{\begin{smallmatrix} R(abc) \\ R(abc') \end{smallmatrix}\right\}$, follows $\left\{\begin{smallmatrix} y=c \\ y=c' \end{smallmatrix}\right\}$.

Second proof (using only III, IV, and V): Take x, α, and y so that $R(bx\alpha)$, $R(xb\alpha)$, and $R(bay)$. Then from $R(bx\alpha)$, $\left\{\begin{smallmatrix} R(abc) \\ R(abc') \end{smallmatrix}\right\}$, $R(xb\alpha)$, $R(bay)$, follows $\left\{\begin{smallmatrix} y=c \\ y=c' \end{smallmatrix}\right\}$. In either case, $c = c'$.

In § 6: *Proof of theorems* 1–2. In order to show that either half of postulate V' is sufficient, the proof is divided into four steps, (a–d):

a) Proof of theorem 1, using $V'(1)$: Given $R(abc)$ and $R(a'bc)$: to prove, $a = a'$. Take z so that $R(za'a)$ and y so that $R(cya')$. From $R(za'a)$,

$R(a'bc)$, $R(abc)$, follows $R(zcc)$; and from $R(zcc)$, $R(cya')$, $R(cya')$, follows $R(za'a')$. Hence, from $R(za'a)$ and $R(za'a')$, we have $a = a'$.

b) Proof of $V'(2)$ from $V'(1)$ and theorem 1: Given, $R(abp)$, $R(bcq)$, and $R(aqN)$. Take P so that $R(PcN)$ and A so that $R(AbP)$. Then from $R(AbP)$, $R(bcq)$, $R(PcN)$, follows $R(AqN)$. From $R(AqN)$ and $R(aqN)$, we have $A = a$; and from $R(abP)$ and $R(abp)$ we have $P = p$. Hence $R(pcN)$.

c) Proof of theorem 2, using $V'(2)$: Given, $R(abc)$ and $R(ab'c)$; to prove $b = b'$. Take z so that $R(bzb')$ and x so that $R(xcb)$. From $R(abc)$, $R(bzb')$, $R(ab'c)$, follows $R(czc)$; and from $R(xcb)$, $R(czc)$, $R(xcb)$, follows $R(bzb)$. Hence, from $R(bzb)$ and $R(bzb')$, we have $b = b'$.

d) Proof of $V'(1)$ from $V'(2)$ and theorem 2: Given, $R(abp)$, $R(bcq)$, and $R(pcM)$. Take Q so that $R(aQM)$, and C so that $R(bCQ)$. Then from $R(abp)$, $R(bCQ)$, $R(aQM)$, follows $R(pCM)$. From $R(pCM)$ and $R(pcM)$ we have $C = c$; and from $R(bcQ)$ and $R(bcq)$ we have $Q = q$. Hence $R(aqM)$.

Thus from either half of postulate V' we can prove the other half, and hence both the theorems 1–2.

In § 6: *Proof of theorem* 3′. Given a and b; take z so that $R(aza)$, b' so that $R(bb'z)$, and c so that $R(cb'a)$. Then c is the required element such that $R(abc)$. For, take β so that $R(a\beta c)$ and β' so that $R(\beta\beta'z)$. Then from $R(a\beta c)$, $R(\beta\beta'z)$, $R(aza)$, follows $R(c\beta'a)$, by $V'(2)$. From $R(c\beta'a)$ and $R(cb'a)$ we have $\beta' = b'$, by theorem 2; and from $R(\beta b'z)$ and $R(bb'z)$ we have $\beta = b$, by theorem 1. Hence $R(abc)$.

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DEFINITIONS OF A GROUP AND A FIELD BY INDEPENDENT POSTULATES*

BY

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Introduction.

1. The simple definition here given for a general abstract group relates as to origin and character to Professor MOORE's two definitions, Transactions, vol. 3 (1902), pp. 485–492. A few days before the appearance of the addition, Transactions, vol. 5 (1904), p. 549, to his paper, Professor MOORE remarked to me that one of his postulates relating to an inverse was redundant, meaning postulate $(3''_r)$ of his second definition. I thought the reference was to his first definition and attempted to reconstruct the proof of a redundancy, there absent. This attempt led me to alter his postulate (4_l) to read $aa'_r = i_r$ instead of $a'_l a = i_r$ and to note that postulate (3_l) becomes redundant in the altered set, thus obtaining the present definition. Subsequently I learned that Professor MOORE, in his proof of the redundancy of $(3''_r)$ in his second definition, had obtained relations‡ sufficient to establish the present definition but had not applied them to set up the definition itself.

The present postulates for a general group possess the desirable property that they remain independent within sets of postulates for special classes of groups, the specialization being either in the direction of the number $(n > 1)$ of elements or their commutativity (§§ 3–5).

The definition of a field (§ 6), based on the present definition of a group, has evident advantages over the earlier definitions.§

The postulates for a field remain independent under an assumption that the set is finite, or forms an enumerable infinitude, or a non-enumerable infinitude.

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‡ MOORE, this number of Transactions, p. 179.

§ HUNTINGTON and DICKSON, Transactions, vol. 4 (1903), p. 31, p. 13. For the new definitions by HUNTINGTON, with a bibliography, see this number of Transactions, p. 181.

Definition of a group, §§ 2–5.

2. Given a function $a \circ b$ of two arguments and a set of elements, we will say that the elements form a group with respect to $\circ$ when the following postulates hold:*

(1_1) For every two† equal or distinct elements a and b of the set, there is a determination of $a \circ b$.

(1_2) For a and b in the set, there is at most one determination of $a \circ b$.

(1_3) If, for a and b in the set, there is at least one determination of $a \circ b$, one determination is an element of the set.

(2) $(a \circ b) \circ c = a \circ (b \circ c)$ whenever a, b, c, and all the determinations of $a \circ b$, $b \circ c$, $(a \circ b) \circ c$, and $a \circ (b \circ c)$ occur in the set.

(3) There occurs in the set an element i such that, for every element a of the set, $a \circ i$ has the determination a.

(4) If such elements i occur, then for a particular i, and for every‡ a in the set there occurs in the set an element a' such that $a \circ a'$ has the determination i.

Postulates (1_1), (1_2), (1_3) may be combined into the triple statement:

(1) For every two equal or distinct elements a and b of the set $a \circ b$ is uniquely determined as an element of the set.

In view of (3) and (4), to any element a there correspond elements a' and a'' such that $a \circ a' = i$, $a' \circ a'' = i$, where i is a fixed element such that $e \circ i = e$ for every element e. Applying also (1) and (2), we have

$$a = a \circ i = a \circ (a' \circ a'') = (a \circ a') \circ a'' = i \circ a'',$$

$$a' \circ a = a' \circ (i \circ a'') = (a' \circ i) \circ a'' = a' \circ a'' = i.$$

Hence $a' \circ a = i$. By this theorem, $a'' \circ a' = i$. Hence

$$i \circ a = (i \circ i) \circ a = [i \circ (a'' \circ a')] \circ a = (i \circ a'') \circ (a' \circ a) = a \circ i = a.$$

Since $a \circ i = a = i \circ a$ for every a, i is called an identity element. If also i_1 is an identity element, then $i_1 = i_1 \circ i = i$. *Hence there is an unique identity element.* Since $a \circ a' = i = a' \circ a$, a' is called an inverse of a. If also a_1' is an inverse of a, then

$$a_1' = a_1' \circ i = a_1' \circ (a \circ a') = (a_1' \circ a) \circ a' = i \circ a' = a'.$$

Hence there is an unique inverse of each element.

* In (2), (6), and (7), we mean by $A = B$ that one of the determinations of A equals one of the determinations of B.

† The assumption need not be made for $a \circ i$ or $a \circ a'$, in view of (3), (4).

‡ The assumption need not be made for $a = i$, in view of (3).

3. We prove that *the postulates* (1_1), (1_2), (1_3), (2), (3), (4), (5_k), $k=1, 2,$ *or* 3, *are consistent and independent*, where

(5_1) The number of distinct elements is a fixed integer n, $n>1$;*

(5_2) The distinct elements of the set form an enumerable infinitude;

(5_3) The distinct elements form a non-enumerable infinitude.

Their consistency follows from the existence of the group of the elements $0, 1, \cdots, n-1$ with $a \circ b = a+b \pmod n$, and the group of all rational (or real) numbers with $a \circ b = a+b$.

Let I_1 be a set containing i and exactly $n-1$ further elements b; I_2 or I_3 a set containing i and further elements b forming an enumerable or a non-enumerable infinitude, respectively.

To prove the independence of a postulate (j), we exhibit a set $[j]$ in which postulate (j) fails while each of the remaining postulates hold. In the sets, $a \circ b$ is understood to have a unique determination unless the contrary is stated.

$[1_1]$ Set I_k, $i \circ i = i$, $b \circ b = i$, $b \circ i = b$, no determination of $i \circ b$ or of $b \circ b' \ (b \neq b')$.

$[1_2]$ Set I_k forming a group under $\odot$; $a \circ c$ with the determinations $a \odot c$ and i.

$[1_3]$ Set I_k; $i \circ i = i$, $b \circ i = b$, $b \circ b' = i$, $i \circ b$ not in the set.

$[2]$ Set I_k; $i \circ i = i$, $b \circ i = b$, $b \circ b' = i$, $i \circ b = i$.

$[3]$ Set I_k; $i \circ i = i$, $b \circ i = i$, $b \circ b' = i$, $i \circ b = i$.

$[4]$ Set I_k; $i \circ i = i$, $b \circ i = b$, $i \circ b = b$, $b \circ b' = b_1$, b_1 a fixed b.

4. We next examine the effect of adding the commutative law:

(6) $a \circ b = b \circ a$ whenever a, b, and all the determinations of $a \circ b$ and $b \circ a$ occur in the set.

The postulates (1_1), (1_2), (1_3), (2), (3), (4), (5_k), (6), *for* $k=0, 2,$ *or* 3, *are consistent and independent.* Here the new postulate is

(5_0) The number of distinct elements is finite but undetermined.

The proof follows from the sets $[1_1]$, $[1_2]$, $[1_3]$, $[3]$, $[4]$ above, and

$[2]'$ Set I_k, $n>2$; $i \circ i = i$, $i \circ b = b \circ i = b$, $b \circ b = i$, $b \circ b' = b_1 (b \neq b')$, Then, for $b \neq b_1$, $(b \circ b_1) \circ b_1 = b_1 \circ b_1 = i$, $b \circ (b_1 \circ b_1) = b \circ i = b$.

$[6]$ There exist finite and infinite non-commutative groups. †

The question of the independence of the postulates for $k=1$ is answered by § 5 in connection with the sets $[1_1]$, $[1_2]$, $[1_3]$, $[2]'$, $[3]$, $[4]$.

5. THEOREM. ‡ *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, *where* $p_1, \cdots, p_s$ *are distinct primes, and each* $\alpha_j > 0$. *The necessary and sufficient conditions that all existing*

* For $n=1$, the group may be defined by the independent postulates (1_1), (1_2), (1_3); also by the independent postulates (1_2), (3).

† E. g., that of the transformations $x' = \pm x + b$, where b ranges over all integers, or all real numbers, or the residues modulo m, $m>2$.

‡ Addition of February 2, 1905.

groups of order n shall be abelian are: (i) each $\alpha_j \leqq 2$; (ii) no $p_j^{\alpha_j} - 1$ is divisible by one of the primes $p_1, \cdots, p_v$.

We first prove that the conditions are necessary. If $\alpha_1 > 2$, a non-abelian G_n is given by the direct product of the cyclic groups $C_{p_k^{\alpha_k}} (k = 2, \cdots, v)$ and a non-abelian $G_{p_1^{\alpha_1}}$, say of the type generated by P and Q where

$$P^{p_1^{\alpha_1-1}} = I, \qquad Q^{p_1} = I, \qquad Q^{-1}PQ = P^{1+p_1^{\alpha_1-2}} \qquad (\alpha_1 > 2).$$

If each $\alpha_j \leqq 2$ and $p_1^{\alpha_1} - 1$ is divisible by p_2, there exists a non-abelian group of order $p_1^{\alpha_1} p_2^{\alpha_2}$ and hence, as before, one of order n. Indeed, if $\alpha_1 = \alpha_2 = 1$, we use the group generated by P_1 and P_2 where

$$P_1^{p_1} = I, \qquad P_2^{p_2} = I, \qquad P_2^{-1} P_1 P_2 = P_1^{\pi} \qquad [p_1 \equiv 1 \pmod{p_2}],$$

where π is an existing primitive root of $x^{p_2} \equiv 1 \pmod{p_1}$. If $\alpha_1 = 1$, $\alpha_2 = 2$, we use the direct product of the preceding group by a cyclic C_{p_2}. If $\alpha_1 = 2$, $\alpha_2 = 1$, the case in which $p_1 - 1$ is divisible by p_2 is disposed of as before, while for $p_1 + 1$ divisible by p_2, $p_2 > 2$, we use the group* generated by S, T_1, T_2, with

$$T_1^{p_1} = T_2^{p_1} = S^{p_2} = I, \quad T_1 T_2 = T_2 T_1, \quad S^{-1} T_1 S = T_2, \quad S^{-1} T_2 S = T_1^{-1} T_2^{b},$$

where b is an (existing) integer such that $x^2 + bx + 1 \equiv 0 \pmod{p_1}$ is irreducible, viz., $b = i + i^{p_1}$, i a mark of the $GF[p_1^2]$ belonging to the exponent p_2. Finally, if $\alpha_1 = \alpha_2 = 2$, we use the direct product of one of the preceding groups by C_{p_2}.

It remains to prove that the conditions (i) and (ii) are sufficient to make every G_n abelian. To proceed by induction, we assume that this statement is true for every $n' = p_1^{\beta_1} \cdots p_e^{\beta_e}$, $\beta_1 \leqq \alpha_1, \cdots, \beta_e \leqq \alpha_e$, $n' < n$. Hence every subgroup of G_n is abelian, so that† either G_n is abelian or else n is divisible by at most two distinct primes. But for $n = p$, $n = p^2$, or $n = p_1^{\alpha_1} p_2^{\alpha_2}$, $0 < \alpha_1 \leqq 2$, $0 < \alpha_2 \leqq 2$, $p_1^{\alpha_1} \not\equiv 1 \pmod{p_2}$, $p_2^{\alpha_2} \not\equiv 1 \pmod{p_1}$, G_n is immediately seen to be abelian. Hence the induction is complete.

Definition of a field, §§ 6–9.

6. We employ a set of elements and two functions $a \oplus b$ and $a \otimes b$. For $\circ = \oplus$, postulate (j) of §§ 2, 4, with $j = 1, 2, 3, 4$, or 6, is designated j^+. An element i satisfying 3^+ is designated i_+. For $\circ = \otimes$, (j) is designated $j^\times$; but $4^\times$ is assumed only for elements a distinct from each i_+. An element i satisfying $3^\times$ is designated $i_\times$. As the distributive law we take‡

*HÖLDER, Mathematische Annalen, vol. 43 (1893), pp. 345–357. We may use the form by COLE and GLOVER, American Journal of Mathematics, vol. 15 (1893), pp. 207–8.

† MILLER and MORENO, Transactions, vol. 4 (1903), p. 398.

‡ If we alter (7) to read $a \otimes (b \oplus c) = (a \otimes c) \oplus (a \otimes b)$, we deduce 6^+ by taking $a = i_\times$ and applying $6^\times$. Then also (7) follows.

(7) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ whenever a, b, c, and all the determinations of the functions involved occur in the set.

In view of the theorems in §§ 7, 8, we make the definition:* A set of elements forms a field with respect to $\oplus$ and $\otimes$ if postulates 1^+, 2^+, 3^+, 4^+, $1^\times$, $2^\times$, $3^\times$, $4^\times$, $6^\times$, and 7 hold.

7. THEOREM.† *From* 1^+, 2^+, 3^+, 4^+, $1^\times$, $3^\times$, $6^\times$, 7, *follows* 6^+.

We substitute $\beta \oplus \gamma$ for a in (7), apply 1^+, $1^\times$, $6^\times$, 7 and get

$$(\beta \oplus \gamma) \otimes (b \oplus c) = [(b \otimes \beta) \oplus (b \otimes \gamma)] \oplus [(c \otimes \beta) \oplus (c \otimes \gamma)],$$

for every β, γ, b, c in the set. Interchanging b with β, c with γ, and noting that the first member is unaltered in view of $6^\times$, we get by $6^\times$ and 2^+,

$$(b \otimes \beta) \oplus (c \otimes \beta) \oplus (b \otimes \gamma) \oplus (c \otimes \gamma) = (b \otimes \beta) \oplus (b \otimes \gamma) \oplus (c \otimes \beta) \oplus (c \otimes \gamma),$$

Since 1^+, 2^+, 3^+, 4^+ imply an inverse under $\oplus$, we get

$$(c \otimes \beta) \oplus (b \otimes \gamma) = (b \otimes \gamma) \oplus (c \otimes \beta).$$

Taking $\beta - \gamma = i_\times$, we get $c \oplus b = b \oplus c$.

8. THEOREM. *Let* i_+ *be chosen so that* 4^+, *as well as* 3^+, *is satisfied. Then* $a \otimes i_+ = i_+ \otimes a = i_+$ *for every element* a. *If* $a \otimes b = i_+$ *and* $b \neq i_+$, *then* $a = i_+$. *Hence* i_+ *has the ordinary properties of zero under* $\oplus$ *and* $\otimes$.

By 7 for $b = c = i_+$, $e = e \oplus e$, where $e = a \otimes i_+$. Hence (§ 2), $a \otimes i_+ = i_+$ for every a. By $1^\times$, $6^\times$, $i_+ \otimes a = i_+$.

By $2^\times$, from $a \otimes b = i_+$ follows $i_+ = a \otimes (b \otimes c)$ for every c. Let $i_\times$ be chosen so that $4^\times$, as well as $3^\times$, holds. Taking c such that $b \otimes c = i_\times$, we get $i_+ = a \otimes i_\times = a$.

9. THEOREM. *Postulates* 1^+, 2^+, 3^+, 4^+, $1^\times$, $2^\times$, $3^\times$, $4^\times$, $6^\times$, 7, 5_k ($k = 0$, 2, *or* 3) *are independent.*

For $k = 0$, i. e., for a finite number of elements, we employ the sets:

[1^+] Elements $0, 1, -1$; $a \oplus b = a + b$, $a \otimes b = a \times b$.

[2^+] $0, 1, -1$; $0 \oplus a = a \oplus 0 = a$, $a \oplus b = 0 (a \neq 0, b \neq 0)$, $\otimes = \times$.

[3^+] $0, 1, \cdots, n-1 (n > 1)$; $a \oplus b = b$, $a \otimes b = a + b \pmod{n}$.

* If the set is finite, we may omit 3^+ and insert 6^+. Then if 3^+ fails, all the elements form a group with respect to $\otimes$. It cannot contain an element a of period α, $\alpha > 1$. Indeed, if we set $y = i_\times \oplus a \oplus a^2 \oplus \cdots \oplus a^{\alpha-1}$, we get $a \otimes y = y$, whence $a = i_\times$. The single element in the set necessarily satisfies 3^+.

† Some months after devising this proof I learned that a similar proof had been given by HILBERT, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899-1900), p. 183. But earlier writers have noted the essential point in the proof; viz., that the uniqueness of the expansion of $(\beta + \gamma)(b + c)$ depends upon the validity of the commutative law for addition.

[4^+]

$\oplus$	0	1
0	0	1
1	1	1

$\otimes$	0	1
0	0	0
1	0	1

[$1^\times$] $0, 1$; $a \oplus b = a + b \pmod 2$, $0 \otimes a = 0$, $1 \otimes 1 = 1$, $1 \otimes 0$ not in the set.

[2^+] Eight* elements (ξ, η, ζ), ξ, η, ζ taken modulo 2;

$$(\xi, \eta, \zeta) \oplus (x, y, z) = (\xi + x, \eta + y, \zeta + z),$$

$$(\xi, \eta, \zeta) \otimes (x, y, z) = (z\xi + x\zeta, (x + y)\xi + (x + z)\eta + y\zeta, y\xi + x\eta + z\zeta).$$

The latter equals $i_\times = (0, 0, 1)$ if $D \equiv \xi + \zeta + \xi\zeta \not\equiv 0 \pmod 2$ and

$$x \equiv \xi(1 + \zeta)/D, \; y \equiv (\xi + \xi\eta + \eta\zeta)/D, \; z \equiv \zeta(1 + \xi)/D.$$

But $D \equiv 0$ only when $\xi \equiv \zeta \equiv 0$; while if $\xi \equiv \zeta \equiv 0$, $\eta \equiv 1$, we have

$$(0, 1, 0) \otimes (x, y, z) = (0, x + z, x) = i_\times \text{ if } x \equiv z \equiv 1.$$

Finally, $2^\times$ fails for $a = (0, 1, 0)$, $b = (0, 1, 0)$, $c = (1, 1, 1)$.

[$3^\times$] $0, 1$; $a \oplus b = a + b \pmod 2$, $a \otimes b = 0$.

[$4^\times$] $0, 1, \cdots, n - 1, n$ composite; $a \oplus b = a + b \pmod n$, $a \otimes b = a \times b \pmod n$

[$6^\times$] Nine elements $a + bj$, $a, b \equiv 0, 1, -1 \pmod 3$;

$$(a + bj) \oplus (c + dj) = a + c + (b + d)j,$$

$$(a + bj) \otimes (c + dj) = ac - (-1)^{ab} bd + \{bc + (-1)^{ab} ad\} j,$$

where the exponent ab is taken in the form $0, 1, \text{or } -1$. Then $j \otimes j = -1$,

$$j \otimes (1 + j) = -1 + j, \qquad (1 + j) \otimes j = 1 - j,$$

so that $6^\times$ and the right-hand distributive law fail. For $4^\times$,

$$(a + bj) \otimes \left(\frac{a - (-1)^{ab} bj}{a^2 + b^2} \right) = 1 = i_\times,$$

since $a^2 + b \equiv 0$ only when $a \equiv b \equiv 0 \pmod 3$. Computation shows that $2^\times$ holds.

[7] Any finite group of order > 1; $a \oplus b = a \otimes b = a \circ b$.

[5_0] All rational numbers; $\oplus = +$, $\otimes = \times$.

* A much simpler statement of Σ_7, Transactions, vol. 4 (1903), p. 20.

For $k=2\,(k=3)$, *i. e.*, for an enumerable (a non-enumerable) infinitude of elements, we employ the following sets $[j]$, R denoting the set of all rational (real) numbers, R_+ that of all positive rational (positive real) numbers:

$[1^+]$ R; $\otimes = \times$, $a \oplus 0 = 0 \oplus a = a$, $a \oplus a = 0$, $a \oplus b$ not in R if $a \neq b$, $a \neq 0$ or $b \neq 0$.

$[2^+]$ R; $\otimes = \times$, $a \oplus a = 0$, $a \oplus b = a + b\,(a \neq b)$.

$[3^+]$ R_+; $\oplus = +$, $\otimes = \times$.

$[4^+]$ R_+ and zero; $\oplus = +$, $\otimes = \times$.

$[1^\times]$ R; $\oplus \equiv +$, $a \otimes b = ab$ (if $a = 1, b = 1$, or $ab = 1$), otherwise $a \otimes b$ not in R.

$[2^\times]$ Hypercomplex numbers $\alpha + \beta i + \gamma j$, α, β, γ arbitrary in R;

$\otimes$	1	i	j
1	1	i	j
i	i	-1	1
j	j	1	-2

Then $(ii)j = -j$, $i(ij) = i$; $(\alpha + \beta i + \gamma)(x + yi + zj) = 1$ if

$$x = \alpha/\Delta, \quad y = -\beta/\Delta, \quad z = -\gamma/\Delta, \quad \Delta = \alpha^2 + \gamma^2 + (\beta - \gamma)^2.$$

$[3^\times]$ R; $a \oplus b = a + b$, $a \otimes b = 0$.

$[4^\times]$ Positive and negative integers and zero; $\oplus = +$, $\otimes = \times$. (All $a_1 e_1 + a_2 e_2$ with a_1 and a_2 real, $e_1^2 = 0$, $e_1 e_2 = e_2 e_1 = e_1$, $e_2^2 = e_2$; $\oplus = +$, $\otimes = \times$; then $e_1 \otimes y = i_\times = e_2$ is impossible since $e_1(y_1 e_1 + y_2 e_2) = y_2 e_1$.)

$[6^\times]$ All quaternions $ai + bj + ck + d$, a, b, c, d in R; $\oplus = +$, $\otimes = \times$.

$[7]$ R_+; $\oplus = \otimes = \times$.

$[5_k, k = 2 \text{ or } 3]$ Any finite field, *e. g.*, the classes of residues modulo p.

THE UNIVERSITY OF CHICAGO.

ON SEMI-GROUPS AND THE GENERAL ISOMORPHISM BETWEEN INFINITE GROUPS*

BY

LEONARD EUGENE DICKSON†

1. When there exists a correspondence between the elements of two finite groups such that the product of two elements of one corresponds to the product of the corresponding elements of the other, then the elements of either which correspond to the identity of the other group form themselves a group. It is somewhat surprising that this familiar theorem fails in general for infinite groups, as shown by definite examples in §§ 7–8. Nevertheless, DE SÉGUIER ‡ has attempted to establish the theorem for any groups; the error § in his argument is quite subtile. The correct theorem involves the concept ‖ *semi-group*, which reduces to a group when there is a finite number of elements, but not in general for an infinitude of elements.

2. Given a function $a \circ b$ of two arguments and a set of elements, we say that the elements form a semi-group with respect to $\circ$ when the following postulates hold:

(1) For every two elements a and b of the set, $a \circ b$ is uniquely determined as an element of the set.

(2) $(a \circ b) \circ c = a \circ (b \circ c)$, in the sense of Transactions, p. 199.

(3) If a, x, x' occur in the set, and if there are equal determinations in the set of $a \circ x$, $a \circ x'$, then $x = x'$.

(4) If there are equal determinations of $x \circ a$, $x' \circ a$, then $x = x'$.

We may replace the triple statement (1) by the three postulates (1_1), (1_2), (1_3) of Transactions, p. 199.

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‡ *Éléments de la théorie des groupes abstraits* (1904), p. 66. In this text a group is any finite or infinite group unless a limitation is explicitly given. The theorem in question is expressly applied to infinite groups in § 66; cf. top of p. 68.

§ *Ibid.*, p. 66, l. 9: "tous les éléments ··· répondent d'ailleurs à a_i^{-1}."

‖ Otherwise defined by DE SÉGUIER, *ibid.*, p. 8. Cf. the writer's review in the Bulletin of the American Mathematical Society, vol. 11 (1904), pp. 159–164. The postulates (3) and (4) there given by the writer are here replaced by the equivalent but more desirable postulates (3) and (4). The proof of the independence of the new postulates applies as well to the former, whose independence was stated without proof.

Semi-groups fall naturally into three classes, according as

(5_1) The number of elements is a fixed integer n;

(5_2) The elements of the set form an enumerable infinitude;

(5_3) The elements of the set form a non-enumerable infinitude.

To establish the independence of the postulates of the system

$$S_k = (1, 2, 3, 4, 5_k) \qquad (k = 1, 2, \text{or } 3),$$

with $n > 2$ in (5_1), we exhibit, for $j = 1, 2, 3, 4, 5_k$, a set Σ_j of elements and a rule of combination $\circ$ such that the jth postulate fails while the remaining four hold. According as $k = 1, 2$, or 3, we employ as

Σ_1: N, R with 2 omitted, P with 2 omitted, with $a \circ b = a + b$;

Σ_2: C; R, with $a \circ b = a^{-1} \times b^{-1}$; P, with $a \circ b = a^{-1} \times b^{-1}$;

Σ_3: N, R, P, with $a \circ b = a$;

Σ_4: N, R, P, with $a \circ b = b$;

Σ_{5k}: R, P, R, with $a \circ b = a \times b$;

where N is the set of the first n positive integers, R the set of all positive rational numbers, P the set of all positive real numbers; and where C is any group of order n modified by interchanging two rows of its multiplication-table; for example, having the elements $0, 1, \cdots, n-1$, such that for any element b,

$$0 \circ b = (b+1), \qquad 1 \circ b = b, \qquad a \circ b = (a+b) \text{ if } a > 1,$$

(d) denoting the least positive residue of d modulo n. Then

$$(0 \circ 0) \circ 0 = 1 \circ 0 = 0, \qquad 0 \circ (0 \circ 0) = 0 \circ 1 = 2.$$

A finite semi-group is a group. Indeed, the system* S_1 is precisely the definition of a finite group given by WEBER.† The corresponding definition of a finite commutative group was given earlier by KRONECKER.‡ Every infinite group is a semi-group, but not inversely.

3. THEOREM. *A set of elements belonging to a group and such that the product of any two occurs in the set forms a semi-group.*

* Not until I had written out the independence proof for S_1 did I notice that a similar proof had been given by HUNTINGTON at the end of his paper in the Bulletin of the American Mathematical Society, vol. 8 (1902), pp. 296–300.

† Mathematische Annalen, vol. 20 (1882), p. 302; *Lehrbuch der Algebra*, 2d ed., vol. 2, pp. 3–4.

‡ KRONECKER, Berliner Monatsbericht, 1870, pp. 882–883.

Indeed, postulates (1)–(4) hold for the set. For example, the elements $a^n (n = 1, 2, \cdots)$ of the infinite cyclic group $\{a, a^{-1}\}$ form a semi-group.

4. THEOREM. *In a semi-group any right-hand identity is a left-hand identity, and conversely; there is at most one identity.*

If i is a right-hand identity, then for every a and b,

$$a \circ (i \circ b) = (a \circ i) \circ b = a \circ b,$$

whence $i \circ b = b$ by (3), so that i is a left-hand identity. The converse follows from $(b \circ i) \circ a = b \circ (i \circ a) = b \circ a$ and (4). Finally, if i and i_1 are identities, $i = i \circ i_1 = i_1$.

5. THEOREM. *In a semi-group containing the identity, any right-hand inverse of a is a left-hand inverse of a, and conversely; there is at most one inverse of a.*

If $a \circ a' = i$, then

$$a \circ (a' \circ a) = (a \circ a') \circ a = i \circ a = a = a \circ i,$$

whence $a' \circ a = i$ by (3). The converse follows similarly from (4).

6. A correspondence between the elements of two sets A and B is called *mutual* when, if a is one of the elements corresponding to b, b is one of the elements corresponding to a. We write $a \sim b$.

Two groups A and B are called *isomorphic* if there exists a mutual correspondence between their elements such that to each element of A corresponds one or more elements of B and to each element of B corresponds one or more elements of A, and such that $a_i a_j \sim b_i b_j$ if $a_i \sim b_i$, $a_j \sim b_j$.

Let A' denote the set of elements of A corresponding to the identity I_B of B, and B' the set of elements of B corresponding to I_A.

If $a_i' \sim I_B$, and $a_j' \sim I_B$, then $a_i' a_j' \sim I_B$. From § 3 follows the

THEOREM. *Each of the sets A' and B' is a semi-group.*

7. As an example in which neither of the semi-groups A' and B' is a group, consider the two infinite cyclic groups

$$A = \{a, a^{-1}\}, \qquad B = \{b, b^{-1}\}. \tag{6}$$

To a^i we make correspond b^{-i+j} $(j = 0, 1, 2, \cdots)$, for every integer i positive, negative, or zero. Then to $a^i a^l$ correspond

$$b^{-(i+l)+r} = b^{-i+j} b^{-l+k} \qquad (r = j + k = 0, 1, 2, \cdots).$$

Moreover for any integer i, it follows that to b^i correspond a^{-i+j} $(j = 0, 1, 2, \cdots)$ and no further elements of A. Thus

$$A' = (I, a, a^2, \cdots), \qquad B' = (I, b, b^2, \cdots),$$

neither being a group. But the correspondence obeys the definition in § 6.

8. As an example in which neither A' nor B' contains the identity, consider groups (6) and the correspondences

$$I_A \sim b^{1+j}, a \sim b^j, a^{2i} \sim b^{-i+j}, a^{2i+1} \sim b^{-i+j}, a^{-i} \sim b^{i+j} \quad (j=0, 1, 2, \cdots),$$

i being any positive integer. Then, inversely,

$$I_B \sim a^{1+j}, b^i \sim a^{-i+j}, b^{-i} \sim a^{2i+j} \qquad (j=0, 1, 2, \ldots),$$

while any element of B corresponds only to the indicated elements of A. The correspondence obeys the definition in § 6.

9. THEOREM. *If one of the semi-groups A' and B' is a group, the other is.*

Suppose that B' is a group. Let a' be any element of A', and let b be one of the elements of B which correspond to a'^{-1}. Since $I_B \sim a'$, then $bI_B \sim a'^{-1}a' = I_A$. Hence b is in B'. Then b^{-1} lies in the group B'. Hence

$$b^{-1} \sim I_A, I_B = b^{-1}\, b \sim I_A\, a'^{-1} = a'^{-1},$$

so that a'^{-1} lies in A'. Hence A' is a group.

COROLLARY I. If A' or B' includes only a finite number of elements, A' and B' are groups.

COROLLARY II. If the groups A and B are isormorphic in the usual special sense so that there exists a mutual correspondence between their elements such that to each element of A corresponds an unique element of B and to each element of B corresponds one or more elements of A, and such that $a_i a_j \sim b_i b_j$ if $a_i \sim b_i$, $a_j \sim b_j$, then the elements of A which correspond to the identity of B form a group.

THE UNIVERSITY OF CHICAGO,
November 30, 1904.

A SET OF POSTULATES FOR ORDINARY COMPLEX ALGEBRA*

BY

EDWARD V. HUNTINGTON

Introduction.

The well-known algebra which forms one of the main branches of elementary mathematics is a body of propositions expressible in terms of five fundamental concepts—(the class of complex numbers, with the operations of addition and of multiplication, and the subclass of real numbers, with the relation of order)—and deducible from a small number of fundamental propositions, or hypotheses.

The object of the present paper is to analyze these fundamental propositions, as far as may be, into their simplest component statements, and to present a list which shall not only be free from redundancies, and sufficient to determine the algebra uniquely, but shall also bring out clearly the relative importance of the several fundamental concepts in the logical structure of the algebra.

A more precise statement of the problem is the following: we consider two undefined classes, K and C; two undefined operations, which we may denote by $\oplus$ and $\odot$; and an undefined relation, which we may denote by $\circledcirc$; and we impose upon these five (undefined) fundamental concepts certain arbitrary conditions, or postulates, to serve as the fundamental propositions of an abstract deductive theory (the other propositions of the theory being all the propositions which are deducible from the fundamental propositions by purely logical processes); the problem then is, to choose these fundamental propositions so that all the theorems of algebra, regarded as formal or abstract propositions, shall be deducible from them—the class K and C corresponding to the classes of complex and real numbers, respectively, and the symbols $\oplus$, $\odot$, and $\circledcirc$ to the ordinary $+$, $\times$, and $<$.

Furthermore, the set of postulates, to be satisfactory, must determine the algebra *uniquely;* in other words, the set of postulates adopted must be such that any two systems $(K, C, \oplus, \odot, \circledcirc)$ which satisfy them all shall be *simply isomorphic* with respect to the fundamental concepts—that is, shall be capable of being brought into one-to-one correspondence in such a way that correspond-

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ing operations performed on corresponding elements shall lead to corresponding results (see the detailed theorem in § 8).*

The ordinary algebra will appear, then, as one among many *equivalent* concrete interpretations of the abstract theory — all the possible interpretations being called *equivalent, with respect to the fundamental concepts* K, C, $\oplus$, $\odot$, *and* $\ominus$, because they are not distinguishable by any properties which can be stated in terms of these symbols.† (Illustrations of such equivalent, or abstractly identical, systems will be given below in § 10.)

Finally, for the sake of elegance, the postulates should be *independent;* that is, no one of them should be deducible from the rest.

A problem of this kind can be solved, no doubt, in a great variety of ways; in fact, one has considerable freedom, not only in the choice of the postulates themselves, but also in the choice of the fundamental concepts in terms of which the postulates are stated. For example, in the present paper, the class C might easily be defined in terms of K, $\oplus$, $\odot$, and $\ominus$, instead of being introduced as a fundamental concept.‡ Or again, it would be possible, I believe, to develop the whole algebra on the basis of a single fundamental operation, in terms of which C, $\oplus$, $\odot$, and $\ominus$ could all be defined.§

For the present purpose, however, I have not attempted to reduce the number of fundamental concepts to a minimum, but have sought to give a set of postulates which shall conform as closely as possible to familiar forms of presentation. With this end in view, I have omitted the circles from the symbols $\oplus$, $\odot$ and $\ominus$, wherever it is possible to do so without confusion with the ordinary $+$, $\times$, and $<$ of arithmetic; it must be constantly borne in mind, however, that the

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*A set of postulates having this property has been called a *categorical* set, as distinguished from a *disjunctive* set; see O. VEBLEN, Transactions, vol. 5 (1904), p. 346. The notion of equivalence, which had long been familiar in the case of two isomorphic groups, became in the hands of G. CANTOR the fundamental notion of his theory of classes (Mengenlehre, théorie des ensembles).

†In the case of any categorical set of postulates one is tempted to assert the theorem that if any proposition can be stated in terms of the fundamental concepts, either it is itself deducible from the postulates, or else its contradictory is so deducible; it must be admitted, however, that our mastery of the processes of logical deduction is not yet, and possibly never can be, sufficiently complete to justify this assertion. My statement in the last footnote on page 17 of the present volume of the Transactions must therefore be taken, as Mr. H. N. DAVIS first pointed out to me, with some qualification. Compare in this connection remarks by D. HILBERT in his address on the problems of mathematics at the Paris congress of 1900, translated in the Bulletin of the American Mathematical Society, ser. 2, vol. 8 (1901–02), especially pp. 444–445; also his "Axiom of Completeness" for real numbers in the Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899), p. 183.

‡To do this, we have merely to demand the existence of a class C having all the properties mentioned in the postulates of groups II–V, below. The additional postulate required for this purpose would be thus in a high degree a compound statement.

§The operation I have in mind is the operation of taking "the absolute value of the difference of."

symbols $+$, $\times$, and $<$, when used in this more general sense, are simply arbitrary signs, having no properties not expressly stated in the postulates.

Furthermore, in selecting the postulates, I have chosen only such statements as do not involve the assumption of the existence of any kind of numbers.

It should be said, in conclusion, that no attempt is here made to give a metaphysical analysis of the concepts *class*, *operation*, and *relation*, on which the algebra is based, or of the *laws of deductive logic* by which its propositions are deduced — the discussion of these more fundamental notions, here assumed as familiar, being matter for the trained student of philosophy. I hope, however, that a paper like the present may be indirectly of service on the philosophical side of the subject, by enabling one to formulate very precisely the problems involved in the question: *What is algebra?*

I have made use, in the course of the work, of results obtained in my previous paper on real algebra, which will be found in the present volume of the Transactions.* To this paper, and to the *Theoretische Arithmetik* of STOLZ and GMEINER (1901–), as well as the *Principles of Mathematics*, by BERTRAND RUSSELL (vol. 1, 1903), the reader is referred for bibliographical references. Compare also the addresses by JOSIAH ROYCE† and MAXIME BÔCHER‡ at the Congress of Arts and Science, St. Louis, September, 1904, and a series of articles by LOUIS COUTURAT in a French review.§

PART I. THE POSTULATES FOR COMPLEX ALGEBRA, AND DEDUCTIONS FROM THEM.

We consider *two classes*, K and C, *two operations*, $+$ and $\times$, and a *relation*, $<$, subject to the conditions prescribed in the twenty-eight postulates of this section. ‖ The symbols $a + b$ and $a \times b$ (or ab) may be read as the "sum" and the "product" of the elements a and b, by analogy with the ordinary language of arithmetic, while the relation $a < b$, or $b > a$, may be read "a pre-

* E. V. HUNTINGTON, *A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups*, Transactions, vol. 6 (1905), pp. 17–41. — The statement on page 21, that the postulates $R1$–$R8$ form a categorical set, is clearly erroneous, and is corrected in §4 below. Since this statement was merely parenthetical, the correction does not affect the rest of the paper. — In postulate $R6$, on pages 20 and 32, the element x in 2°) must be noted as "different from X," in order to make the proof of independence for $R3$ conclusive.

† J. ROYCE, *The sciences of the ideal*, published in Science for October 7, 1904 (new series, vol 20, pp. 449–462).

‡ M. BÔCHER, *The fundamental conceptions and methods of mathematics*, published in the Bulletin of the American Mathematical Society, vol. 11 (1904-05), pp. 115–135.

§ L. COUTURAT, *Les principes des mathématiques*, in the Revue de Métaphysique et de Morale, vol. 12, 1904.

‖ A summary of these postulates will be given below, in §9.

cedes b," or "b follows a."* As already noted, however, these interpretations of the symbols are by no means the only possible ones (compare the examples below, in § 10).

The postulates are arranged in five groups, numbered I–V.

§ 1. *The class K with regard to $+$ and $\times$.*

The following thirteen postulates, I:1–13, make the class K a *field*† with respect to the operations $+$ and $\times$.

The first seven postulates state the general laws of operation in the field, and are to be understood as holding only in so far as the elements, sums, and products involved are elements of K. The remaining six postulates give the requisite "existence-theorems."

POSTULATE I,1. $a + b = b + a$. (Commutative law for addition.)

POSTULATE I,2. $(a + b) + c = a + (b + c)$. (Associative law for addition.)

POSTULATE I,3. *If* $a + b = a + b'$, *then* $b = b'$.

This may be called the (left-hand) law of cancelation for addition.

POSTULATE I,4. $ab = ba$. (Commutative law for multiplication.)

POSTULATE I,5. $(ab)c = a(bc)$. (Associative law for multiplication.)

POSTULATE I,6. *If* $ab = ab'$, *and* $a + a \neq a$, *then* $b = b'$.

This may be called the (left-hand) law of cancelation for multiplication.

POSTULATE I,7. $a(b + c) = (ab) + (ac)$.

This is the (left-hand) distributive law for multiplication with respect to addition.

POSTULATE I,8. *If a and b are elements of K, then their "sum," $a + b$, is an element of K*; that is, there is an element s, uniquely determined by a and b, such that $a + b = s$.

POSTULATE I,9. *There is an element x in K, such that $x + x = x$.*

From postulates I:1,2,3,8, it follows that there cannot be more than one element z such that $z + z = z$, and by postulate I,9 there must be at least one such element. This unique element z is called the *zero-element* of the system, and will be denoted by 0, whenever there is no danger of confusion with the ordinary 0 of arithmetic.

From the same postulates we have: $0 + a = a + 0 = a$, for every element a. (For proof of this and the preceding statement see p. 23.)

* The expressions "before" and "after," etc., are preferable to the expressions "greater" and "less," etc., in this connection, since the notion of *size* is not involved in the notion of *order*.

† Cf. E. V. HUNTINGTON, *Note on the definitions of abstract groups and fields*, with bibliography, in the present number of the Transactions.

POSTULATE I,10. *If there is a unique zero-element* 0, *then for every element* a *there is an element* a' *such that* $a + a' = 0$.

From postulates I:1,3,10, it follows that a' is uniquely determined by a, and that $a + a' = a' + a = 0$. This element a' is called the *negative of* a, and is denoted by $-a$, so that

$$a + (-a) = (-a) + a = 0.$$

Furthermore, every two elements a and b determine uniquely an element x such that $a + x = b$; this element x, namely $x = (-a) + b$, is called their *difference*, b *minus* a, and is denoted by $b - a$, so that

$$a + (b - a) = b.$$

(The proof depends on postulates I:1,2,3,8,9,10.) The usual properties of subtraction follow readily from this definition. *

POSTULATE I,11. *If* a *and* b *are elements of* K, *then their "product,"* ab, *is an element of* K; that is, there is an element p, uniquely determined by a and b, such that $ab = p$.

POSTULATE I,12. *If there is a unique zero-element* 0 *in* K, *then there is an element* y *in* K *different from* 0, *and such that* $yy = y$.

From postulates I:4,5,6,11, we see that there cannot be more than one element u, different from 0, such that $uu = u$, and by postulate I,12 there must be at least one such element. This unique element u is called the unit-element of the system, and will be denoted by 1 whenever there is no danger of confusion with the ordinary 1 of arithmetic.

From the same postulates we have: $1 \times a = a \times 1 = a$, for every element a. (Proof: if $uu = u$, then $u(ua) = (uu)a = ua$, whence $ua = a$, by I,6.)

POSUTLATE I,13. *If there is a unique zero-element* 0, *and a unique unit-element* 1, *different from* 0, *then for every element* a, *provided* a *is not* 0, *there is an element* a'' *such that* $aa'' = 1$.

From postulates I:4,6,13, it follows that a'' is uniquely determined by a, and that $aa'' = a''a = 1$ (where $a \neq 0$). This element a'' is called the *reciprocal of* a, and is denoted by $\frac{1}{a}$, or $1/a$, so that

$$a(1/a) = (1/a)a = 1 \text{ if } a \neq 0.$$

Furthermore, every two elements a and b, provided $a \neq 0$, determine uniquely an element y such that $ay = b$; this element y, namely $y = (1/a)b$, is called

* It would probably be preferable, however, on both pedagogical and logical grounds, not to introduce the operation of subtraction at all, but to regard $b - a$ always as the sum of the two elements b and $-a$.

their *quotient, b divided by a*, and is denoted by $\frac{b}{a}$, or b/a, so that

$$a(b/a) = (b/a)a = b \text{ if } a \neq 0.$$

(The proof depends on postulates I:4,5,6,11,12,13.) The usual properties of division follow readily from this definition.*

Now from the distributive law (postulate I,7), which we have not yet utilized, we see that

$$0 \times a = a \times 0 = 0,$$

for every element a. Then if a is different from 0, its reciprocal, $1/a$, will also be different from 0, since otherwise we should have $1 = 0$. Hence the important theorem that *if a and b are both different from* 0, *their product is also different from* 0; or in other words, a product ab cannot be zero unless at least one of its factors is zero. [Proof: if $ab = 0$ and $a \neq 0$, then $b = 1 \times b = (1/a)a \times b = (1/a) \times ab = (1/a) \times 0 = 0$.]

These theorems are sufficient to show that any system $(K, +, \times)$ which satisfies the postulates I:1–13 has all the properties of a *field* with respect to the operations $+$ and $\times$. To summarize:

The elements of K form an abelian group with respect to $+$; *the elements excluding the zero of that group form an abelian group with respect to* $\times$; *for every element* a, $a \times 0 = 0 \times a = 0$; *and the operation* $\times$ *is distributive with respect to the operation* $+$.

We now consider the multiples, submultiples, and rational fractions of any element of the field.

Assuming the system of the ordinal, or natural, numbers, the characteristic properties of which are summarized in PEANO's five postulates (see page 27), we define the *mth multiple, ma, of any elememt a* by the usual recurrent formulæ

$$1a = a,\ 2a = 1a + a,\ 3a = 2a + a,\ \cdots,\ (k+1)a = ka + a,$$

where $1, 2, 3, \cdots, k, \cdots, m$ denote ordinal numbers, and $k + 1$ the arithmetical sum of k and 1.

The *mth submultiple of a*, denoted by a/m, is then defined by the equation $a/m = a/(mu)$,† from which we have $m(a/m) = a$.

From these definitions it follows at once that $p(a/q) = (pa)/q$, so that we may denote either member of this equation by pa/q.

* It would probably be preferable, however, on both pedagogical and logical grounds, not to introduce the operation of division at all, but to regard b/a always as the product of the two elements b and $1/a$.

† The unit-element of the field is here denoted by u, instead of by 1, to avoid confusion with the number 1.

Any element of the form pa/q (where p and q are ordinal numbers) is called a *rational fraction of* a. The following theorems on rational fractions in which m, n, p, q denote ordinal numbers, are readily deduced from the properties of a field:

$$(a)\quad \frac{mpa}{mq} = \frac{pa}{q}; \qquad (b)\quad \frac{pa}{q} + \frac{ma}{n} = \frac{(np+mq)a}{qn}; \qquad (c)\quad \frac{pa}{q} \times \frac{ma}{n} = \frac{pma}{qn}.$$

It should be noticed that we cannot yet infer from $ma = na$ that $m = n$, since, as far as postulates I:1–13 are concerned, the field may be finite, that is, contain only a finite number of elements.* (See below, under postulate IV, 2.)

§ 2. *The subclass C, with regard to + and ×.*

The following seven postulates, II:1–7, make C a subclass in K, which shall be, like K, a field with respect to the operations + and ×.

POSTULATE II,1. *If a is an element of C, then a is an element of K.*

POSTULATE II,2. *The class C contains at least one element.*

POSTULATE II,3. *If a is an element of C, then there is an element b in C, such that* $a \neq b$.

These three postulates tell us that C is a subclass in K, containing at least two elements.

POSTULATE II,4. *If a and b are elements of C, then their sum,* $a + b$, *if it exists at all in K, is an element of C.*

POSTULATE II,5. *If a is an element of C, then its negative,* $-a$, *if it exists at all in K, is an element of C.*

Hence the zero element, 0, of K is an element of C.

POSTULATE II,6. *If a and b are elements of C, then their product,* ab, *if it exists at all in K, is an element of C.*

POSTULATE II,7. *If a is an element of C, then its reciprocal,* $1/a$, *if it exists at all in K, is an element of C.*

Hence the unit element, 1, of K is an element of C.

Now by virtue of postulate II,1, all the general laws of operation, I:1–7, hold as well for the subclass C as for the whole class K; hence the postulates of I and II make the subclass C, like K, a *field* with respect to + and ×.

It should be noticed that the subclass C may be identical with K, as far as the postulates of I and II are concerned. (See § 6.)

§ 3. *The subclass C, with regard to* $<$.

In the following five postulates, III:1–5, we impose further restrictions on the subclass C, which, together with the postulates II:2–3, make the class C a *one-*

* As is well known, the number of elements in a finite field must be a prime number, or a power of a prime.

dimensional continuum (in the sense defined by DEDEKIND) with respect to the relation $\prec$. (On the use of the word continuum, see remarks in §4.) The fifth postulate, III,5, however, proves to be a consequence of the other four, when the properties of the field (I and II), and the postulate IV:1–2 are utilized; it should therefore not be included in the final list.

POSTULATE III,1. *If a and b are elements of C, and $a \neq b$, then either $a \prec b$ or else $a \succ b$.*

POSTULATE III,2. *If $a \prec b$, then $a \neq b$.*

That is, the relation $\prec$ is *non-reflexive* throughout C.

POSTULATE III,3. *If a, b and c are elements of C, and if $a \prec b$ and $b \prec c$, then $a \prec c$.*

That is, the relation $\prec$ is *transitive* throughout C.

From III:2–3 we see that the relations $a \prec b$ and $a \succ b$ cannot both be true;* that is, the relation $\prec$ is *non-symmetric* for every pair of elements in C. Hence, *if a and b are elements of C, we must have either*

$$a = b, \text{ or } a \prec b, \text{ or } a \succ b;$$

and these three relations are mutually exclusive.

POSTULATE III,4. *If Γ is a non-empty subclass in C, and if there is an element b in C such that every element of Γ precedes b, then there is an element X in C having the following two properties with regard to the subclass Γ:*

1°) *every element of Γ precedes X or is equal to X; while*

2°) *if x' is any element of C which precedes X, then there is at least one element of Γ which follows x'.*

This is the postulate of *continuity* in the form essentially due to DEDEKIND.†

The element X, which is readily seen to be uniquely determined by the subclass Γ, is called the "upper limit" of Γ, or sometimes its "least upper bound." From III:1–4 can be deduced, as on page 20, the following theorem, which is DEDEKIND's original form of the postulate of continuity:

If the elements of C are divided into two non-empty classes, Γ and Γ', such that every element of C belongs either to Γ or to Γ', and if every element of Γ precedes every element of Γ', then there is an element X in C which has the following properties:

1°) *every element of C which precedes X belongs to Γ; and*

2°) *every element of C which follows X belongs to Γ'.*

It should be noticed in regard to this postulate of continuity, so-called, that all the postulates III:1–4 might be satisfied by a *discrete* assemblage; in order to introduce the property of *density*, we add the following postulate, III,5:

* In my previous paper (§1) this proposition, at least for the case $a \neq b$, was used as a postulate, the other postulates being "weakened" to allow of its independence. It has not seemed worth while to carry the analysis so far in the present article.

† Loc. cit., below.

Postulate III,5. If a and b are elements of C, and $a < b$, then there is at least one element x in C, such that $a < x$ and $x < b$. Such elements x are said to lie *between* a and b.

These five postulates, III:1–5, together with postulates II:2–3 (which merely tell us that the class C contains at least two elements), make the class C a *one-dimensional continuum*, in DEDEKIND's sense, with respect to the relation $<$ (see § 1 in my previous paper).

The postulate III,5 is not printed in italics, because, as already stated, when we utilize the postulates of I, II and IV, as well as those of III, it can be shown to be redundant. The proof will be given in § 5.

§ 4. *Note on the use of the word "continuum."*

The term *one-dimensional* (or *linear*) *continuum* with respect to the relation of order ($<$), as used in this and my previous paper, must be understood not in the exact sense defined by CANTOR* in 1895, but in the older and wider meaning in which the word was used by DEDEKIND† and WEBER.‡

DEDEKIND would call any system $(K, <)$ which has at least two elements and satisfies the postulates III:1–5, a continuum.

CANTOR restricts the term to systems which possess not only all these properties, but also two further properties, namely: 1) the class K shall have a first and a last element; and 2) the class K *shall contain a denumerable subclass*, R, *such that between every two distinct elements of* K *there shall be at least one element of* R.§ (A denumerable class means an infinite class which can be placed in one-to-one correspondence with the class of natural numbers.)‖ He

*G. CANTOR, *Beiträge zur Begrundung der transfiniten Mengenlehre*, Mathematische Annalen, vol. 46 (1895), p. 510; French translation by F. MAROTTE, *Sur les fondements de la théorie des ensembles transfinis*, 1899, p. 42. Cf. B. RUSSELL, loc. cit., chap. 36, and L. COUTURAT, loc. cit., pp. 664–675.

† R. DEDEKIND, *Stetigkeit und irrationalen Zahlen*, 1872; English translation by BEMAN.

‡ H. WEBER, *Algebra*, vol. 1 (1898), pp. 4–12.

§ Compare also A. N. WHITEHEAD, in a footnote on p. 299 of B. RUSSELL's *Principles of Mathematics*, vol. 1 (1903).

‖ The term *abzählbar*, introduced by G. CANTOR in 1879 [Mathematische Annalen, vol. 15 (1879), p. 4; vol. 20 (1882), p. 116; vol. 21 (1883), p. 550], and uniformly translated in French by *dénombrable* [Acta Mathematica, vol. 2 (1883), p. 365; Formulaire de Mathématiques, vol. 4 (1903), p. 130], has been variously rendered in English. The Century Dictionary gave *enumerable* in 1889; but the editor in charge of the mathematical vocabulary, Mr. C. S. PEIRCE, has since used *numerable*, and finally *denumerable*, in his own writings [see Monist, vol. 2, p. 541, 1892; and especially vol. 7, p. 214, 1897]. *Enumerable* is also the form proposed by Professor OSGOOD [American Journal of Mathematics, vol. 19, p. 161, 1897], and by Professor BEMAN [Bulletin of the American Mathematical Society, vol. 3, p. 174, 1897]. At least one writer, W. H. YOUNG, uses *countable* [Proceedings of the London Mathematical Society, vol. 34, p. 285, 1902, and later papers]; while the Encyclopedia Brittanica prefers *numerable* (article "Number," vol. 31, p. 282, 1902; G. B. MAT-

then shows* that any two systems which satisfy this definition of the linear continuum are *ordinally similar*, and equivalent to the class of real numbers from 0 to 1 inclusive, as far as the relation $\prec$ is concerned. His postulates therefore form a *categorical set*, with regard to the relation $\prec$.

DEDEKIND's definition, on the other hand, is not categorical, but admits various non-equivalent types of order. This possibility is due not merely to the ambiguity in regard to the presence or absence of end points, but rather to the omission of CANTOR's second requirement, concerning the existence of a denumerable subclass within the system. This remarkable postulate is independent of all the other properties, as may be shown by the following example:†

Let K = the class of all complex numbers (α, β) in which α and β are real numbers such that $0 \leqq \alpha \leqq 1$ and $0 \leqq \beta \leqq 1$; with $(\alpha_1, \beta_1) \in (\alpha_2, \beta_2)$ whenever $\alpha_1 \prec \alpha_2$; and if $\alpha_1 = \alpha_2$, then whenever $\beta_1 \prec \beta_2$.

This system (which can be stated, if one prefers, in geometric terms) satisfies all of DEDEKIND's postulates, and has a first element, $(0, 0)$, and a last element, $(1, 1)$; but it is not a continuum under CANTOR's definition, since it is not possible to find within it a denumerable subclass of the kind required.‡

CANTOR's definition is thus clearly superior to DEDEKIND's for the purposes of a purely ordinal discussion of the continuum; if, however, as in algebra, the elements of the system considered are not only connected by the relation of order, but are also combined by the operation of addition,§ then DEDEKIND's definition may be, as it is in this paper, amply sufficient. It is therefore perhaps not undesirable that both definitions should remain in current use.

A further analysis of CANTOR's last postulate, with a general discussion of geometric continuity, will be found in a paper by O. VEBLEN in the present number of the Transactions.

§ 5. *Further postulates for subclass C.*

The following two postulates, IV:1–2, serve to connect the operations $+$ and $\times$ and the relation $\prec$, within the subclass C, and make it equivalent to the class of all real numbers with respect to $+$, $\times$, and $\prec$.

THEWS]. The form *denumerable* is the form adopted by RUSSELL and WHITEHEAD in their *Principles of Mathematics*, and seems to me preferable to the other terms because it avoids a suggestion of finitude (cf. PEIRCE, loc. cit., 1897). I am also following RUSSELL and WHITEHEAD, and many Italian writers, in translating *Menge* (Mannigfaltigkeit, ensemble, insieme, aggregato) by *class* instead of by manifold, mass, set, ensemble, collection, assemblage, or aggregate.

* Loc. cit.

† Cf. H. WEBER, loc. cit., p. 11.

‡ For, there would have to be elements of the subclass for every value of α between 0 and 1, and these values of α form an infinite class which is not denumerable.

§ WEBER, loc. cit., speaks of such systems as systems which admit *measurement* (messbare Mannigfaltigkeiten).

POSTULATE IV,1. *Within the class C, if $x \prec y$, then $a + x \prec a + y$, whenever* $a + x \neq a + y$.*

From this postulate we have at once: if $a \succ 0$ and $b \succ 0$, then $a + b \succ a$; and if $a \prec 0$ and $b \prec 0$, then $a + b \prec a$ (compare postulates $RA1$ and $RA2$ in my previous paper, page 25). Hence, if we define *positive and negative elements* as those elements of C which are $\succ 0$ and $\prec 0$ respectively, we can show that if $a \prec b$ then there is a positive x such that $a + x = b$, and conversely, if x is positive in $a + x = b$, then $a \prec b$. [Proof as for theorem 14 on page 26.]

POSTULATE IV,2. *Within the class C, if $a \succ 0$ and $b \succ 0$, then $ab \succ 0$.*

By the aid of this postulate it is easy to show that the product, ab, of two elements of C will be positive when a and b are both positive or both negative; and negative if a and b are of opposite signs (compare theorem 33 on page 40). Hence the unit-element 1, or u, with all its multiples and submultiples, will be positive;† and therefore all the multiples and submultiples of any positive element a will also be positive.

We are now in a position to prove the theorem of *density*, which was stated provisionally in § 3 as postulate III,5. The proof, which involves all the postulates I–IV, is as follows: If $a \prec b$, $b - a$ will be positive, and if we take $c = (b - a)/2$, c will also be positive; if then we take $x = a + c$ (from which follows $x + c = b$), we find at once that $a \prec x$ and $x \prec b$.

It thus appears that all the postulates of the set for real algebra given in the appendix of my previous paper (page 39) are satisfied by the class C; hence, in any system $(K, C, +, \times, \prec)$ which satisfies the twenty-six postulates of groups I–IV, the subclass C will be equivalent to the class of all real numbers with respect to $+$, $\times$, and $\prec$.

§ 6. *Note on real algebra.*

As we have just seen, any class C which is a field with respect to $+$ and $\times$, and possesses DEDEKIND's property of continuity with respect to $\prec$, and satisfies the postulates IV:1–2, will be equivalent to the class of all real numbers with respect to $+$, $\times$, and $\prec$.

Hence the twenty-six postulates of groups I–IV, with an additional postulate demanding that the class K shall contain no elements which do not belong to C, would form a categorical set of postulates for real algebra.

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* This proviso makes possible the proof of the independence of postulate I,3, which would otherwise be deducible from IV,1, with the aid of III,2.

† For, if we suppose $u \prec 0$, then $-u \succ 0$, whence $u \times (-u) \prec 0$, or $-u \prec 0$, and therefore $u \succ 0$. The multiples of u will form an ascending sequence: $u \prec 2u \prec 3u \prec \cdots$; whence: if $mu = nu$, then $m = n$; or more generally, if $ma = na$, and $a \neq 0$, then $m = n$, where m and n are ordinal numbers.

On the other hand, these same twenty-six postulates, with the addition of the two postulates of group V, below, form a categorical set for complex algebra.

We may say, then, that the postulates of groups I–IV contain the properties common to both real and complex algebra.

§ 7. *Postulates peculiar to complex algebra.*

The two following postulates, V:1–2, which complete the list of postulates for complex algebra, concern the existence of elements of K which do not belong to the subclass C.

POSTULATE V,1. *If K is a field with respect to $+$ and $\times$, then there is an element j in K such that $jj = -1$ (where -1 is the negative of the unit-element of the field).*

If i is one element which has this property (namely, $ii = -1$), then $-i$ will have the same property; and this will be true of no other element besides these two. That is, the element j whose existence is postulated in V.1 is not uniquely determined, but may have two values, one the negative of the other. Which of these values we shall denote by i and which by $-i$ is a matter of arbitrary choice.

Neither i nor $-i$ can be an element of C, since the square of every element of C is positive or zero; hence, if x and y are elements of C, the product iy and the sum $x + iy$ will be elements of K which do not belong to C unless $y = 0$. This gives us the important theorem that if $x + iy = x' + iy'$, where x, y, x', y', belong to C, then $x = x'$ and $y = y'$.*

The next, and last, postulate demands that the class K shall contain no further elements, that is, no elements not expressible in the form $x + iy$, where x and y are elements of C.

POSTULATE V,2. *If K and also C are fields with respect to $+$ and $\times$, and if there is an element i such that $ii = -1$ (see postulate V,1), then for every element a in K there are elements x and y in C such that $x + iy = a$.*

We are now in a position to establish the categorical character of the whole set of postulates.

§ 8. *Proof of the equivalence of all systems that satisfy the postulates* I–V.

The following theorem for complex algebra is analogous to the theorem numbered 37 in my paper on real algebra (see page 40), and is proved in a similar way.

If $(K, C, +, \times, <)$ and $(K', C', +, \times, <)$ are any two systems satisfying the twenty-eight postulates of groups I–V, then these systems are EQUIV-

*Thus, from $x + iy = x' + iy'$ follows $x - x' = i(y' - y)$; hence $y' - y = 0$, and therefore $x - x' = 0$.

ALENT, *or* ABSTRACTLY IDENTICAL, *with respect to* K, C, $+$, $\times$, *and* $\prec$; *that is, the classes* K *and* K' *can be brought into one-to-one correspondence in such a way that if* a', b', *etc., are the elements of* K' *which correspond to the elements* a, b, *etc., in* K, *then we shall have :*

1°) *the subclass* C *corresponds to the subclass* C' ;

2°) *if* $a + b = c$, *then* $a' + b' = c'$, *and conversely ;*

3°) *if* $ab = c$, *then* $a'b' = c'$, *and conversely ;*

4°) *within the subclasses* C *and* C', *if* $a \prec b$, *then* $a' \prec b'$, *and conversely.*

In other words, the twenty-eight postulates form a categorical set.*

The proof of the theorem consists simply in bringing the elements of C in K into correspondence with the elements of C' in K', as in the proof of the analogous theorem for real algebra, and then making the element i in K correspond to the element i' in K'. Then every element $x + iy$ in K will correspond to an element $x' + iy'$ in K', where x' and y' are the elements in C' which correspond to the elements x and y in C.

It should be noticed that this correspondence between two equivalent systems can be set up in two, and only two, ways; the only ambiguity resulting from the arbitrary choice of the element i (see postulate V,2)†.

PART II. SUMMARY OF THE POSTULATES, AND PROOF OF THEIR CONSISTENCY AND INDEPENDENCE.

In this part, for convenience of reference, I give a list of the twenty-eight postulates of groups I–V, using the general symbols $\oplus$, $\odot$, $\circledcirc$, z, and u, instead of the symbols $+$, $\times$, $\prec$, 0, and 1; these latter symbols are used, in this part, only in their ordinary arithmetic meanings.

Next, I give some examples of systems $(K, C, \oplus, \odot, \circledcirc)$ which satisfy all the twenty-eight postulates. These systems, as has just been shown, are all equivalent, or *abstractly identical*, with respect to $\oplus$, $\odot$, and $\circledcirc$; the existence of any one of them proves the *consistency* of the postulates.

Finally, I give, for each of the twenty-eight postulates, an example of a system $(K, C, \oplus, \odot, \circledcirc)$ which satisfies all the other postulates, but not the one in question, thus establishing in the usual way the *independence* of all the postulates.

In constructing these systems, the existence of the ordinary systems of real and complex numbers, as derived by the "genetic" method from the system of the natural numbers, is assumed, as well as the usual geometric representations of these numbers by points of the plane or the sphere.

* See the first two footnotes in the introduction, above.

† Speaking in geometric terms, the complex plane has two *aspects:* the first plane may be rotated through 180° about the axis of reals before being applied to the second plane.

§ 9. *List of the postulates.*

Before giving the postulates themselves, it will be convenient to repeat the more important definitions which have been used in the course of the work. All these definitions are simply abbreviations, introduced in order to avoid tedious circumlocution.

DEFINITION 1. If there is a uniquely determined element z such that $z \oplus z = z$, then z is called the *zero-element*, or zero.

DEFINITION 2. If there is a unique zero-element z (see definition 1), and if there is a uniquely determined element u, different from zero, and such that $u \odot u = u$, then u is called the *unit-element*, or unity.

DEFINITION 3. If there is a unique zero-element z (see definition 1), and if a given element a determines uniquely an element a' such that $a \oplus a' = z$, then a' is called the *negative of* a, and is denoted by $-a$.

DEFINITION 4. If there is a unique zero-element z and a unique unit-element u (see definitions 1 and 2), and if a given element a, different from z, determines uniquely an element a'' such that $a \odot a'' = u$, then a'' is called the *reciprocal of* a, and is denoted by $1/a$.

The first seven postulates, giving the general laws of operation in the system, are to be understood to hold only in so far as the elements, sums, and products involved are elements of K.

POSTULATE I,1. $a \oplus b = b \oplus a$.

POSTULATE I,2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

POSTULATE I,3. If $a \oplus b = a \oplus b'$, then $b = b'$.

POSTULATE I,4. $a \odot b = b \odot a$.

POSTULATE I,5. $(a \odot b) \odot c = a \odot (b \odot c)$.

POSTULATE I,6. If $a \odot b = a \odot b'$, and $a \oplus a \neq a$, then $b = b'$.

POSTULATE I,7. $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

POSTULATE I,8. If a and b are elements of K, then $a \oplus b$ is an element of K.

POSTULATE I, 9. There is an element x in K such that $x \oplus x = x$.

POSTULATE I,10. If there is a unique zero-element z in K (see definition 1), then for every element a in K there is an element a' in K, such that $a \oplus a' = z$.

POSTULATE I,11. If a and b are elements of K, then $a \odot b$ is an element of K.

POSTULATE I,12. If there is a unique zero-element, z, in K (see definition 1), then there is an element y in K, different from z, and such that $y \odot y = y$.

POSTULATE I,13. If there is a unique zero-element, z, and a unique unit-element, u, different from z, in K (see definitions 1 and 2), then for every element a in K, provided $a \neq z$, there is an element a'' in K such that $a \odot a'' = u$.

The postulates I:1–13 *make the class* K *a field with respect to* $\oplus$ *and* $\odot$.

POSTULATE II,1. If a is an element of C, then a is an element of K.

POSTULATE II,2. The class C contains at least one element.

POSTULATE II,3. If a is an element of C, then there is an element b in C, such that $a \neq b$.

POSTULATE II,4. If a and b are elements of C, then $a \oplus b$, if it exists in K at all, is an element of C.

POSTULATE II,5. If a is an element of C, then its negative, $-a$ (see definition 3), if it exists in K at all, is an element of C.

POSTULATE II,6. If a and b are elements of C, then $a \odot b$, if it exists in K at all, is an element of C.

POSTULATE II,7. If a is an element of C, then its reciprocal $1/a$ (see definition 4), if it exists in K at all, is an element of C.

The postulates II:1–7, *taken with the postulates* I:1–13, *make the sub-class* C, *like the class* K, *a field with respect to* $\oplus$ *and* $\odot$.

POSTULATE III,1. If a and b are elements of C, and $a \neq b$, then either $a \circledcirc b$ or else $a \circledcirc b$.

POSTULATE III,2. If $a \circledcirc b$, then $a \neq b$.

POSTULATE III,3. If a, b, and c are elements of C, and if $a \circledcirc b$ and $b \circledcirc c$, then $a \circledcirc c$.

POSTULATE III,4. If Γ is a non-empty subclass in C, and if there is an element b in C such that $\alpha \circledcirc b$ for every element α of Γ, then there is an element X in C having the following two properties with regard to the sub-class Γ:

1°) if α is an element of Γ, then $\alpha \circledcirc X$ or $\alpha = X$; while

2°) if x' is any element of C such that $x' \circledcirc X$, there is an element ξ in Γ such that $\xi \circledcirc x'$.

The postulates III:1–4 *and* II:2–3, *taken with the redundant postulate* III,5 (*which is here omitted*), *make the sub-class* C *a one-dimensional continuum with respect to* $\circledcirc$, *in the sense defined by* DEDEKIND.

POSTULATE IV,1. If a, x, y, $a \oplus x$, and $a \oplus y$ are elements of C, and $x \circledcirc y$, then $a \oplus x \circledcirc a \oplus y$, whenever * $a \oplus x \neq a \oplus y$.

POSTULATE IV,2. If a, b, and $a \odot b$ are elements of C, and $a \circledcirc z$ and $b \circledcirc z$, then $a \odot b \circledcirc z$ (where z is the zero-element of definition 1).

The twenty-six postulates of groups I–IV *make the sub-class* C *equivalent to the class of all real numbers with respect to* $\oplus$, $\odot$, *and* $\circledcirc$.

POSTULATE V,1. If K is a field with respect to $\oplus$ and $\odot$, then there is an element j in K such that $j \odot j = -u$, where $-u$ is the negative of the unit-element of the field (see definitions 2 and 3).

* Cf. footnote under postulate IV,1, above.

POSTULATE V,2. If K and also C are fields with respect to $\oplus$ and $\odot$, and if there is an element i such that $i \odot i = -u$ (see postulate V,1), then for every element a in K there are elements x and y in C such that $x \oplus (i \odot y) = a$.

These twenty-eight postulates make the class K equivalent to the class of all (ordinary) complex numbers with respect to $\oplus$, $\odot$, and $\circledcirc$. (See the theorem in § 8.)

§ 10. *The consistency of the postulates.*

To prove that these postulates are consistent—that is, to prove that contradictory propositions can never be deduced from them by the processes of formal logic—it is sufficient to show the existence of any system $(K, C, \oplus, \odot, \circledcirc)$ in which all the postulates are satisfied; for then the postulates themselves and all their logical consequences express properties of this system, and must therefore be free from contradiction (since no really existent system can have contradictory properties).*

The following systems, all of which are *equivalent*, or abstractly identical, with respect to K, C, $\oplus$, $\odot$, and $\circledcirc$, are examples of systems which satisfy all the postulates:

1) K = the class of ordinary complex numbers (that is, the class of all couples of the form (x, y), where x and y are real numbers), with $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$; C = the class of real numbers (that is, the class of DEDEKIND'S "cuts," or the class of CANTOR'S "fundamental sequences"), with $\circledcirc$ defined as the ordinary $<$. Here $z = 0$, $u = 1$, and $j = \pm\sqrt{-1}$.

2) K = the class of all the points in a plane (the ordinary "complex plane"), with $\oplus$ and $\odot$ defined in the usual geometric manner;† C = the points of a fixed straight line in the plane (the "axis of reals"), with $\circledcirc$ defined as "on the left of." Here z is the point $(0, 0)$, u the point $(1, 0)$, and j the point $(0, 1)$ or $(0, -1)$.

* This statement, of course, makes large assumptions in regard to the objective validity of our processes of logical deduction. In this connection see the second of the problems proposed by D. HILBERT at the Paris congress of 1900 (loc. cit.) and an article by A. PADOA in L'Enseignement Mathématique, vol. 5 (1903), pp. 85–91.

† Thus, the point $a + b$ is the point reached by starting from a and travelling a path equal in length and direction to the path from O to b; while the point $a \times b$ is the point whose "angle" (from OX) equals the *sum* of the angles of a and b, and whose "distance" (from O) equals the *product* of the distances of a and b. Here the product, x, of the two distances, α and β, may be defined geometrically by the proportion $x : \alpha = \beta : 1$; or it may be defined in terms of measurement, as follows: let

$$\alpha = \lim_{n=\infty} \left(\frac{p_n}{q_n}\right) \quad \text{and} \quad \beta = \lim_{n=\infty} \left(\frac{r_n}{s_n}\right),$$

where the p_n/q_n and r_n/s_n are successive rational approximations to the lengths of α and β; then the sequence of products $p_n r_n / q_n s_n$ will have a limit, and this limit will be the required product of α and β. [Cf. E. V. HUNTINGDON, Strassburg dissertation, *Die Grund-Operationen an absoluten und complexen Grössen in geometrischer Behandlung*, Braunschweig, 1901.]

System (1) is an "arithmetical" system, of which (2) is the "geometrical representation."

In the following systems, the "complex numbers" employed may be interpreted in either way, the geometric phraseology being perhaps the most convenient. All these systems are obtained from (1) or (2) by a projective transformation of the plane, as explained below.

3) $K =$ the class of ordinary complex numbers; $a \oplus b = a + b$; $a \odot b = kab$, where k is any real number not zero; $C =$ the class of real numbers, with $\circledcirc$ defined as $<$ or as $>$, according as k is positive or negative.

Here $z = 0$, $u = 1/k$, and $j = \pm (1/k)\sqrt{-1}$.

4) $K =$ the class of ordinary complex numbers; $a \oplus b = ab/(a + b)$, except that $a \oplus b = a + b$ whenever a or b or $a + b$ is zero; $a \odot b = ab$; $C =$ the class of real numbers, with $\circledcirc$ defined as the ordinary $\prec$, except that $(a \circledcirc b) = (a \succ b)$ whenever a and b are both positive or both negative.

Here $z = 0$, $u = 1$, and $j = \pm \sqrt{-1}$.

It will be noticed that the ordinary meaning of addition is preserved in system (3), and that of multiplication in system (4).

5) $K =$ the class of ordinary complex numbers; $a \oplus b = a + b - h$, and $a \odot b = kab - hk(a + b) + h(1 + hk)$, where h and k are any real or complex numbers, provided $k \neq 0$;* $C =$ the class of complex numbers whose corresponding points, in the complex plane, lie on the straight line through the two points h and $(1/k) + h$; with $\circledcirc$ defined as the relation of order along this line, the forward direction being so chosen that $h \circledcirc (1/k) + h$.

Here $z = h$, $u = (1/k) + h$, $-u = -(1/k) + h$, and $j = \pm (\sqrt{-1}/k) + h$.

6) Let α, β, γ, δ be any real or complex numbers such that $\alpha\delta - \beta\gamma \neq 0$; and let F, for the moment, stand for the class comprising all the ordinary complex numbers together with an extra element to be denoted by ω. (Geometrically speaking, F is the class of all the points on the complex sphere. including the "North Pole," ω.)† We then define K, $\oplus$, $\odot$, C and $\circledcirc$ as follows:

$K =$ all the elements of F, *excluding one point*, P, where if $\gamma \neq 0$, $P = \alpha/\gamma$, and if $\gamma = 0$, $P = \omega$.

If $a \neq \omega$ and $b \neq \omega$, then

$$a \oplus b = \frac{(2\alpha\gamma\delta - \beta\gamma^2)ab - \alpha^2\delta(\alpha + b) + \alpha^2\beta}{\gamma^2\delta ab - \beta\gamma^2(a + b) + 2\alpha\beta\gamma - \alpha^2\delta},$$

except when the denominator is zero, in which case $a \oplus b = \omega$.

If $a \neq \omega$ and $b \neq \omega$, then

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* For an especially simple case, take $h = -1$ and $k = 1$.

† The complex sphere, omitting the North Pole, is the stereographic projection of the complex plane.

$$a \odot b = \frac{(\beta\gamma^2 + \alpha\delta^2)ab - (\alpha\beta\gamma + \alpha\beta\delta)(a+b) + (\alpha^2\beta + \alpha\beta^2)}{(\gamma\delta^2 + \gamma^2\delta)ab - (\beta\gamma\delta + \alpha\gamma\delta)(a+b) + (\beta^2\gamma + \alpha^2\delta)},$$

except when the denominator is zero, in which case $a \odot b = \omega$.

If $a = \omega$, or $b = \omega$, or both, then $a \oplus b$ and $a \odot b$ are defined as the limits of the expressions above, when ω is replaced by $1/x$ and x approaches 0; it being understood that (on the sphere)

$$\lim_{x=0}\left[\frac{1}{x}\right] = \omega.$$

With these definitions of $\oplus$ and $\odot$, the class K proves to be a field* in which z and u have the following values:

$$\text{if } \delta \neq 0,\ z = \frac{\beta}{\delta}; \qquad \text{if } \delta = 0,\ z = \omega;$$

$$\text{if } \gamma + \delta \neq 0,\ u = \frac{\alpha + \beta}{\gamma + \delta}; \qquad \text{if } \gamma + \delta = 0,\ u = \omega.$$

We then take $C =$ the class of all the points on the circumference of a circle drawn on the sphere through the three points P, z, and u, excluding the point P;† the relation $\circlearrowleft$ being defined as the relation of order along this curve in the direction $P - z - u - P$.

[This system (6), which includes the preceding systems as special cases, was suggested to me by Professor C. L. Bouton, who had noticed that the general formulæ

$$a \oplus b = f[f^{-1}(a) + f^{-1}(b)],$$

and

$$a \odot b = f[f^{-1}(a) \times f^{-1}(b)],$$

in which f and its inverse f^{-1} are single-valued functions, provide a pair of operations which satisfy, in general, the postulates for a field with respect to $\oplus$ and $\odot$. In the present example,

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta} \qquad (\alpha\delta - \beta\gamma \neq 0).$$

This transformation (which is the general projective transformation) carries the system (1) into the system (6), the points 0, 1, and the excluded point ω in system (1) going over into the points z, u, and the excluded point P in system (6).

To obtain the system (3), use $f(x) = x/k$. To obtain (4), take $f(x) = 1/x$, and afterwards denote the point ω by the symbol 0 (the excluded point being here $P = 0$). To obtain (5), use the transformation $f(x) = (x/k) + h$.]

* It is especially important to notice that the values of $a \oplus b$ and $a \odot b$, as given by the definitions, are determinate, and belong to the class K, whenever a and b are elements of K.

† It should be noticed that under the given conditions no two of these three points can coincide.

§ 11. *The independence of the postulates.**

The independence of the postulates is proved by the following twenty-eight systems, in each of which K, $\oplus$, $\odot$, C, and $\circledcirc$ are so defined as to satisfy the twenty-seven other postulates, but not the one in question. No one of the postulates, then, can be deducible from the others; for if it were, every system which had the other properties would have this property also, which is not the case.†

For I,1. K = all positive real numbers; $a \oplus b = b$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

For I,2. K = all real numbers; $a \oplus b = (a + b)/3$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

For I,3. K = all real numbers; $a \oplus \mathrm{b} = 0$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

For I,4. K = all real numbers; $a \oplus b = a + b$; $a \odot b = b$; $C = K$; $\circledcirc = <$.

For I,5. K = the class of all couples (α, β) in which α and β are real numbers; with

$$(\alpha_1, \beta_1) \oplus (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, -\alpha_1\beta_2 - \alpha_2\beta_1);$$

C = the class of real numbers, with $\circledcirc = <$. (If we represent these couples as points $\alpha + i\beta$ in the complex plane, $\oplus$ will be the ordinary addition of complex numbers, and $\odot$ will be the ordinary multiplication followed by reflection in the axis of α.)

For I,6. The same system as for I,5, except that here $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\alpha_2 + \alpha_1\beta_2 + \beta_1\beta_2)$.

For I,7. K = all real numbers; $a \oplus b = a + b$; $a \odot b = a + b + 1$; $C = K$; $\circledcirc = <$.

For I:8–13. In constructing the proof-systems for these postulates, $\oplus$, $\odot$, and $\circledcirc$ are defined as the ordinary $+$, $\times$, and $<$ (except as to $\odot$ in the case of I,11), and the subclass C is taken identical with K; class K itself being defined appropriately for each case, as follows:

8) K = the class of the three real numbers -1, 0, and 1.

9) K = all positive real numbers.

10) K = all positive real numbers with 0.

11) K = all real numbers; $a \odot b = ab$ when $ab = 0$ or 1; otherwise $a \odot b$ not in the class.

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*I am greatly indebted to Mr. G. D. BIRKHOFF, of the Harvard Graduate School, who has read this section in manuscript, and checked the correctness of the proofs of independence.

†This is the method of proving independence which has become familiar during the last decade, especially through the work of PEANO, PADOA, PIERI, and HILBERT; but the remark made in the first footnote in § 10, concerning the proof of consistency, applies also here.

12) $K =$ all even integers.

13) $K =$ all integers.

For II,1. $K =$ all complex numbers of the form (x, y), where x and y are rational; $C =$ all real numbers; with $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$, and $<$.

For II:2–7. In constructing the proof-systems for these postulates we take $K =$ all complex numbers, and define $\oplus$ and $\odot$ as the ordinary $+$ and $\times$; the subclass C and the relation $\circledcirc$ are then chosen appropriately for each postulate, as follows:

2) $C =$ an empty class.

3) $C =$ the class containing the single element 0; with $0 \circledcirc 0$ false.

4) $C =$ the class of all real numbers x such that $-1 \leqq x \leqq 1$; with $\circledcirc = <$.

5) $C =$ all positive real numbers, with or without zero; $\circledcirc = <$.

6) $C =$ all pure imaginary numbers, with 0; $\circledcirc$ defined so that $a \circledcirc b$ is true whenever $a/i < b/i$ (where $i = \sqrt{-1}$).

7) $C =$ all integers, or all even integers; $\circledcirc = <$.

For III:1–3. As the proof-systems for these postulates, take $K =$ all complex numbers, and $C =$ all real numbers, with $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$, and with $\circledcirc$ defined appropriately for each case, as follows:

1) $a \circledcirc b$ always false.

2) $a \circledcirc b$ always true within C.

3) $a \circledcirc b$ always true within C when $a \neq b$; $a \circledcirc a$ false.

For III,4. $K =$ all complex numbers of the form (x, y), where x and y are rational; $C =$ all rational real numbers; and $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$, and $<$.

For IV,1. $K =$ all complex numbers; $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$; $C =$ all real numbers, with $\circledcirc$ defined as the ordinary $<$, except that $(a \circledcirc b) = (a > b)$ whenever a and b are both positive.

For IV,2. $K =$ all complex numbers; $a \oplus b = a + b$; $a \odot b = -ab$; $C =$ all real numbers, with $\circledcirc$ defined as the ordinary $<$.

For V,1. $K =$ all real numbers; $C = K$; $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$ and $<$.

For V,2.* $K =$ the class of all expressions T of the form

$$T = A_m t^m + A_{m+1} t^{m+1} + A_{m+2} t^{m+2} + \cdots,$$

where t is a parameter, and m any integer (positive, negative or zero), while the A's are ordinary complex numbers. The operations $\oplus$ and $\odot$ are defined as the ordinary $+$ and $\times$ for such (finite or infinite) expressions. The class C is

*The systems for V,2 were suggested by the non-archimedean number-system given by D. HILBERT in his *Grundlagen der Geometrie*, 1899, § 33. In this connection compare also H. HANKEL, *Theorie der complexen Zahlensysteme*, 1867, § 31.

the class of all those elements T in which all the coefficients are zero except A_0, and A_0 is real; that is, $C =$ the class of real numbers. Within the class C the relation $\leqslant$ is defined as the ordinary $<$.

This system contains the system of ordinary complex numbers as a part of itself, just as the system of the ordinary complex numbers contains the system of real numbers. A still more inclusive system, still satisfying all the postulates except V,2, may be constructed by replacing the A's in the expression above by expressions S_m, S_{m+1}, S_{m+2}, $\cdots$, of the form

$$S = B_n s^n + B_{n+1} s^{n+1} + B_{n+2} s^{n+2} + \cdots,$$

where s is another parameter, and n any integer, while the B's are ordinary complex numbers.

The list of proofs of independence is thus complete.

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ON IMPRIMITIVE LINEAR HOMOGENEOUS GROUPS*

BY

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1. The present paper is devoted first to the proof of a theorem fundamental in the construction of imprimitive linear homogeneous groups † in a given number of variables. Then, by means of this and earlier theorems given by the author on the subject of linear groups, † JORDAN's theorem, ‡ to the effect that the order of a linear homogeneous group G in n variables is of the form λf, where f is the order of an abelian self-conjugate subgroup of G, and λ is less than a fixed number depending only upon n, is proved for imprimitive groups, a number being found that λ must divide. Finally, the principal imprimitive collineation-groups in 4 variables are found and their generating substitutions given.

THEOREM. *Either an imprimitive linear homogeneous group G can be written in monomial form,* § *or the n variables of the group can be so selected that they fall into k sets of imprimitivity of m variables each ($n = km$), permuted according to a permutation-group K in k letters, which group is transitive (in the sense of transitivity of permutation-groups). The subgroup (G') of G, corresponding to the subgroup (K') of K which leaves one letter unchanged, is primitive (in the sense used in linear homogeneous groups) in the m variables of the set corresponding to the letter that K' leaves unchanged.*

In order that G may be transitive (as a linear homogeneous group, i. e., "irreducible"), it is plainly necessary that its sets of imprimitivity contain the same number of variables, and that the permutation-group K, permuting these sets, is transitive (as a permutation-group). We shall prove that, if the

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† See articles, cited below as *Linear groups* I and II, by the author in these Transactions, vol. 4 (1903), p. 387, and vol. 5 (1904), p. 310, for definitions of terms and phrases used and for theorems employed.

‡ Journal für Mathematik, vol. 84 (1878), p. 89. JORDAN does not find a superior limit to λ. Such a limit is given for *primitive groups* by the author in *Linear groups* II. Dr. J. SCHUR has given a limit for λf for such groups in n variables, the sum of the multipliers of the substitutions of which belong to a given algebraical field (Berliner Sitzungsberichte, January, 1905).

§ This term is used by MASCHKE in American Journal of Mathematics, vol. 17 (1895), p. 168. The author used the word "semi-canonical" in *Linear groups* II, p. 313.

subgroup G' of G, corresponding to the subgroup K' of K leaving one letter fixed, is not primitive in the variables of the set corresponding to the letter that K' leaves fixed, then will G be imprimitive in a greater number of sets than k.

Let the k sets of G be $x_{11}, x_{12}\cdots x_{1m}$; $x_{21}, \cdots, x_{2m}$; $\cdots$; $x_{k1}, \cdots, x_{km}$, and let the corresponding letters of K be denoted by $y_1, y_2, \cdots, y_k$. Let K_i be the subgroup of K, leaving y_i fixed, and let G_i be the corresponding subgroup of G. The group K, being transitive, must contain a substitution (A_2') changing y_1 into y_2, one (A_3') changing y_1 into y_3, etc. The group G is plainly generated by G_1 and $k-1$ substitutions $(A_2, A_3, \cdots)$ found among those of G corresponding to $A_2', A_3', \cdots$, respectively. The group obtained by erasing in G_i all the variables except $x_{i1}, x_{i2}, \cdots, x_{im}$ will be denoted by X_i.

It may readily be seen that, no matter how the group X_1 be written, the variables $x_{i1}, x_{i2}, \cdots, x_{im}$ $(i = 2, 3, \cdots, k)$ may be so selected that the substitutions $A_2, A_3, \cdots$, have the forms

$$\begin{aligned} A_2\colon\quad & x_{11}' = x_{21},\ x_{12}' = x_{22},\ \cdots,\ x_{1m}' = x_{2m};\\ & x_{21}' = \alpha_1 x_{11} + \alpha_2 x_{12} + \cdots + \alpha_m x_{km},\ \text{etc.};\\ A_3\colon\quad & x_{11}' = x_{31},\ x_{12}' = x_{32},\ \cdots,\ x_{1m}' = x_{3m};\ \text{etc., etc.},\\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Let P' be any substitution of K, and P any substitution of G taken from those corresponding to P'. If P' replaces y_i by y_j, the variables $x_{i1}, \cdots, x_{ik}$ are transformed by P in the following manner:

$$P\colon \begin{cases} x_{i1}' = p_{11}x_{j1} + p_{12}x_{j2} + \cdots + p_{1m}x_{jm},\\ x_{i2}' = p_{21}x_{j1} + p_{22}x_{j2} + \cdots + p_{2m}x_{jm},\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_{im}' = p_{m1}x_{j1} + p_{m2}x_{j2} + \cdots + p_{mm}x_{jm}. \end{cases}$$

Now, if the group X_1 is not primitive, the variables $x_{11}, x_{12}, \cdots, x_{1m}$ may be supposed to have been selected so that they fall into $k_1 > 1$ sets $x_{11}, \cdots, x_{1\alpha}$; $x_{1\alpha+1}, \cdots, x_{1\beta}$; $\cdots$, all the variables of each set being by X_1 transformed into linear functions of the variables of the same set, or all into the variables of another set. Then, by building the substitution $A_i P A_j^{-1}$,* belonging to G_1:

$$\begin{aligned} & x_{11}' = p_{11}x_{11} + p_{12}x_{12} + \cdots + p_{1m}x_{1m},\\ & x_{12}' = p_{21}x_{11} + p_{22}x_{12} + \cdots + p_{2m}x_{1m},\\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ & x_{1m}' = p_{m1}x_{11} + p_{m2}x_{12} + \cdots + p_{mm}x_{1m},\\ & \text{etc.; etc.,} \end{aligned}$$

* For A_1 we may take the identical substitution of G.

and bearing in mind the assumption made with regard to X_1, we see that the km variables of G fall into $kk' > k$ sets

$$x_{i1}, \cdots, x_{i\alpha}; \; x_{i\alpha+1}, \cdots, x_{i\beta}; \; \cdots; \qquad (i=1, 2, \cdots, k),$$

which are mutually permuted by P. The variables of G are therefore broken up into a greater number of sets of imprimitivity than k. Starting with these kk'_0 sets and proceeding as above, we conclude that we must arrive ultimately at a selection of imprimitive sets for which the groups X_i are primitive or reduce to groups in one variable each, in which case G is written in monomial form. The theorem stated above is therefore proved.

It may be remarked that the writer's theorem 9, in *Linear groups* II, p. 313, follows immediately after it has been proved that any group, whose order is the power of a prime, is not primitive.

2. We shall now prove JORDAN'S theorem for imprimitive groups. Let us consider such a group G of order g in $n = km$ variables, the group X_i being primitive in the variables $x_{i1}, x_{i2}, \cdots, x_{im}$, if $m > k$. By § 12 of *Linear groups* II, the order of such a group X_i in m variables is of the form $\lambda_i f_i$, where f_i is the order of a self-conjugate subgroup of X_i composed of similarity substitutions, and where λ_i is a factor of a certain number that can always be calculated when m is given. Let us call this number $\phi(m)$.

The subgroup H of G corresponding to the identical substitution of K has for order h, an integral multiple of $g/k!$. This group has a subgroup F composed of substitutions which are similarity-substitutions for each of the groups $X_1, X_2, \cdots, X_k$, and whose order is an integral multiple of $h/\{\phi(m)\}^k$; i. e. an integral multiple of

$$\frac{g}{k!\,\{\phi(m)\}^k}.$$

The group F of order f is abelian, and is evidently invariant within G, and the order of the latter is of the form λf, where λ is a factor of $k!\{\phi(m)\}^k$.

3. We now pass on to the construction of the principal imprimitive collineation-groups in four variables. According to the theorem stated above, unless such a group can be written in monomial form, it must possess two sets of imprimitivity, say (x, y) and (z, u). Only the latter class of groups shall be considered here.

Let G be such a group. It is generated by an intransitive group G':

$$G': \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

where X_1 and X_2 are primitive groups in the variables (x, y) and (z, u) respectively, and a substitution R of the form

$$R: \quad x' = a_1 z + b_1 u, \quad y' = c_1 z + d_1 u, \quad z' = a_2 x + b_2 y, \quad u' = c_2 x + d_2 y.$$

Exhibiting R in the form

$$R: \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \equiv \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ c_2 & d_2 & 0 & 0 \end{bmatrix},$$

we find that $RG'R^{-1}$ takes the form

$$RG'R^{-1}: \begin{pmatrix} PX_2P^{-1} & 0 \\ 0 & QX_1Q^{-1} \end{pmatrix}.$$

It follows that X_1 and X_2 (as collineation-groups in two variables) are transformable one into the other by the matrices P and Q. Also, if A_1 and A_2 are corresponding substitutions of X_1 and X_2 respectively, so are PA_2P^{-1} and QA_1Q^{-1}. Moreover, we may replace R by the substitution

$$R' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot R = \begin{pmatrix} 0 & A_1P \\ A_2Q & 0 \end{pmatrix} = \begin{pmatrix} 0 & P' \\ Q' & 0 \end{pmatrix}.$$

Thus, if P is a matrix belonging to X_1, we may assume that the matrix of P' is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Bearing these things in mind, we can construct the required groups without any theoretical difficulties, though the process will involve some labor, especially in reducing the different types obtained to certain standard forms.

We begin by determining all the groups G' possible. The groups X_1 (or X_2) are the well known tetrahedral, octahedral, and icosahedral groups, and are given in WEBER'S *Algebra*, II, 2d edition, pp. 272–287.

The types sought are generated by the following substitutions (ρ being a factor of proportionality):

	$\rho x'$	$\rho y'$	$\rho z'$	$\rho u'$	
S_1	x	$-y$	z	$-u$	
S_2	y	x	u	z	
$S_3^{(k)}$	$x+y$	$i(-x+y)$	$k(z+u)$	$ik(-z+u)$	$k^{3n}=1.$
$S_4^{(n)}$	x	y	αz	αu	α a primitive root of $\alpha^n=1.$
S_5	ix	$-iy$	z	u	I
S_6	y	$-x$	z	u	
S_7	x	y	iz	$-iu$	
S_8	x	y	u	$-z$	
$S_9^{(k)}$	$x+y$	$i(-x+y)$	$(1+i)kz$	$(1+i)ku$	$k^{3n}=1.$
$S_{10}^{(l)}$	$(1+i)lx$	$(1+i)ly$	$z+u$	$i(-z+u)$	$l^{3n}=1.$
$S_{11}^{(n)}$	x	iy	βz	$i\beta u$	$\beta^n=1.$
S_{12}	x	iy	γz	γu	$\gamma^{4n}=(-1)^n.$
S_{13}	ϵx	$\epsilon^4 y$	ϵz	$\epsilon^4 u$	ϵ a primitive root of $\epsilon^5=1.$
S_{14}	$\epsilon(\omega x+y)$	$x-\omega y$	$\epsilon(\omega z+u)$	$z-\omega u$	$\omega=\epsilon+\epsilon^4.$
S_{15}	ϵx	$\epsilon^4 y$	$\epsilon^2 z$	$\epsilon^3 u$	
S_{16}	$\epsilon(\omega x+y)$	$x-\omega y$	$\epsilon^4(z-\omega u)_1$	$-\epsilon^2(\omega z+u)$	
S_{17}	ϵx	$\epsilon^4 y$	z	u	

and are as follows:

Group.	Order.	Generating Substitutions.
a	$12n$	$S_1, S_2, S_3^{(k)}, S_4^{(n)}.$
b	$4.12.2n$	$S_3^{(k)}, S_4^{(2n)}, S_5, S_6, S_7, S_8.$
c	$12.12.2n$	$S_4^{(2n)}, S_5, S_6, S_7, S_8, S_9^{(k)}, S_{10}^{(l)}.$
d	$24n$	$S_1, S_2, S_3^{(1)}, S_4^{(n)}, S_{11}^{(2n)}.$
e	$4.24.2n$	$S_3^{(1)}, S_4^{(2n)}, S_5, S_6, S_7, S_8, S_{11}^{(4n)}.$
f	$12.24.2n$	$S_4^{(2n)}, S_5, S_6, S_7, S_8, S_9^{(1)}, S_{10}^{(1)}, S_{11}^{(4n)}.$
g	$24.24.2n$	$S_4^{(2n)}, S_5, S_6, S_7, S_8, S_9^{(1)}, S_{10}^{(1)}, S_{11}^{(4n)}, S_{12}.$
h	$60n$	$S_4^{(n)}, S_{13}, S_{14}.$
h''	$60n$	$S_4^{(n)}, S_{15}, S_{16}.$
j	$60.60.2n$	$S_4^{(n)}, S_{13}, S_{14}, S_{17}.$

4. In all of these cases, the groups X_1 and X_2 (considered as collineation-groups in 2 variables) are identical. Noting what was stated at the beginning of § 3 concerning the substitution R, we see then that the matrices P and Q must in the cases a, b, c leave invariant a given tetrahedral group, and

must therefore belong to the octahedral group containing this group self-conjugately. Then, by replacing R by $R' = A \cdot R$, where A is some one of the substitutions of G', we find readily that P may be assumed to be of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$. Under these assumptions, there will be no trouble in determining Q. In all the remaining cases we may take P in the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. After the resulting groups have been reduced to certain standard types, the different substitutions R will be as follows:

$$R_1: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = x, \quad \rho u' = y;$$

$$R_2: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = x, \quad \rho u' = iy;$$

$$R_3: \quad \rho x' = z, \quad \rho y' = iu, \quad \rho z' = x, \quad \rho u' = iy;$$

$$R_4: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = y, \quad \rho u' = -x.$$

We obtain the following types of imprimitive collineation-groups in 4 variables that cannot be written in monomial form:

Group.	Order.	Generators.
1°	$2.12n$	R_1 and a
2°	$2.12n$	R_3 and a $(k=1)$
3°	$2.4.12.2n$	R_1 and b $(k=1)$
4°	$2.4.12.2n$	R_3 and b
5°	$2.12.12.2n$	R_1 and c $(k=l=1)$
6°	$2.12.12.2n$	R_3 and c
7°	$2.24.n$	R_1 and d
8°	$2.4.24.2n$	R_1 and e
9°	$2.12.24.2n$	R_1 and f
10°	$2.12.24.2n$	R_2 and f
11°	$2.24.24.2n$	R_1 and g
12°	$2.60n$	R_1 and h'
13°	$2.60n$	R_4 and h''
14°	$2.60.60.2n$	R_1 and j

The groups 1° and 12° are intransitive if $n=1$. The groups 2° and 7° are intransitive if $n=1$, $k=1$ and $\beta=1$, and 2° is of type 7° if $n=2$ and $k=1$. This is readily seen if new variables $x_1 = x+z$, $y_1 = y+u$, $z_1 = x-z$, $u_1 = y-u$ be chosen. The four groups 1°, 2°, 7°, 12° have the invariant $xu - yz = 0$ and occur therefore among those determined by GOURSAT in Annales scientifiques de l'École Normale supérieure, ser. 3, vol. 6 (1889), pp. 9–102. The corresponding types are there numbered XIV, XVI, XVIII, XV, XIX.

A classification of the imprimitive ("decomposable") groups which are "regular," i. e., possess the bilinear invariant $x_1y_2 - x_2y_1 - z_1u_2 + z_2u_1$, has been given by AUTONNE in Journal de Mathématiques, ser. 5, vol. 7 (1901), p. 351–394, where he gives also an extensive discussion of some of their geometrical properties.